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THE INFLUENCE OF ROTATION ON THE FREE OSCILLATIONS OF THE EARTH†

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CONTENTS

We pursue an abstract investigation of the theory of the infinitesimal free elastic-gravitational oscillations of a fairly general rotating Earth model. By considering in some detail the transition to the non-rotating case, we are able to delineate certain of the intrinsic effects of rotation on the normal mode eigensolutions, and to show how profoundly rotation alters the fundamental mathematical and physical properties of these eigensolutions. In particular, we show that the displacement eigenfunctions of a rotating Earth model are not mutually orthogonal, and that the corresponding normal modes of oscillation cannot in general be represented by pure standing waves. We consider the excitation of the normal modes of oscillation of a rotating Earth model by a transient imposed body force distribution, and we show that the complex dynamical amplitude of each normal mode may, in many geophysical applications, be determined separately, in spite of the lack of orthogonality among the displacement eigenfunctions. The calculation of the associated static response after the decay of the normal modes of oscillation is, on the other hand, complicated considerably by the absence of

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orthogonality. We specifically examine the influence of rotation on the zero-frequency rigid body translational and rotational modes of any non-rotating Earth model, and show how to account for the corresponding rigid body modes of any rotating Earth model in excitation calculations.

1. Introduction

Theoretical investigations of the free elastic-gravitational oscillations of the Earth commonly neglect the fact that the equilibrium state of the Earth is one of nearly steady diurnal rotation. Such an approximation performs two valuable functions. First, it implies the absence of centrifugal forces in the static, equilibrium state; it is then consistent, and useful, to idealize the Earth in equilibrium as having perfect spherical symmetry. This in turn makes it very convenient to use surface spherical harmonics as basis functions in the elastic-gravitational equations of motion. Secondly, and more important, it implies the absence of both centrifugal and Coriolis forces in the dynamical equations. The absence of any velocity-dependent forces in the governing equations renders the theory of the infinitesimal elastic-gravitational oscillations of a non-rotating Earth model simply a special case of the classical theory of the infinitesimal oscillations of any conservative or nearly conservative system. This makes immediately available a wide range of elegant and powerful results concerning such systems (see, for example, Rayleigh 1894; Whittaker 1937).

Most attempts to account for the influence of the Earth's steady rotation have been concerned solely with the computation of the perturbing effect of rotation on the degenerate eigenfrequencies of a non-rotating spherically symmetric Earth model (see, for example, Backus & Gilbert 1961; Luh 1974). Much less attention has been given to the manner in which rotation might modify such classical results as orthogonality relations among eigenfunctions, equipartition of energy during a normal mode of oscillation, and the excitation of free oscillations by externally imposed forces.

We intend, in this paper, to pursue an abstract, systematic investigation of the properties of the isentropic elastic-gravitational free oscillations of a general rotating, self-gravitating, perfectly elastic Earth model. The aim of this investigation is to provide a framework for future more quantitative discussions of the effects of rotation on the free oscillations of the Earth. We first review the essential features of the elastic-gravitational oscillations of a non-rotating Earth model, primarily to serve as a guide in examining the complications which are introduced when the model is allowed to rotate. We adopt the rather general philosophy that, in some sense, the number of degrees of oscillatory freedom of a non-rotating elastic-gravitational configuration is not altered when it is allowed to rotate. Each of the well-known properties of the normal modes of oscillation of a non-rotating Earth model will appear as a special case of some corresponding more general property of the rotating case, and we seek to compare and contrast these properties.

The effects of rotation on the infinitesimal oscillations of systems with a finite number of degrees have been extensively studied (see, for example, Thomson & Tait 1879; Lamb 1932; Lyttleton 1953). To a certain extent, some of the results presented here can be viewed as generalizations of these earlier results to continuous systems with a correspondingly infinite number of degrees of freedom.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2. The free oscillations of a non-rotating Earth model: a review

We consider first a model of the Earth which is in mechanical equilibrium while at rest with respect to some inertial frame of reference. Such an Earth model possesses no net angular momentum. We suppose this Earth model to be composed of an isolated, self-gravitating, perfectly elastic continuum occupying a finite, simply connected volume V with surface ∂V. A point or material particle in this static equilibrium configuration will be denoted by its position vector $\boldsymbol{x}$, measured in an inertial reference frame whose origin $\boldsymbol{O}$ coincides with the Earth model's centre of mass. The unit outward normal to ∂V at the point $\boldsymbol{x}$ will be denoted by $\hat{\boldsymbol{n}}(\boldsymbol{x})$.

We shall denote by $\rho_0(\boldsymbol{x})$, $\phi_0(\boldsymbol{x})$, and $\boldsymbol{T}_0(\boldsymbol{x})$, respectively, the mass density scalar field, gravitational potential scalar field, and initial static stress tensor field of the static equilibrium configuration; the initial static stress tensor is not, in general, required to be isotropic. These three fields are related by Poisson's equation

$$\nabla^2\phi_0(\boldsymbol{x}) = 4\pi G\rho_0(\boldsymbol{x}), \tag{1}$$

as well as by the static mechanical equilibrium condition

$$\rho_0(\boldsymbol{x})\,\boldsymbol{\nabla}\phi_0(\boldsymbol{x}) = \boldsymbol{\nabla}\cdot\boldsymbol{T}_0(\boldsymbol{x}). \tag{2}$$

The outer surface ∂V of the Earth model is assumed to be a traction-free surface, i.e.

$$\hat{\boldsymbol{n}}(\boldsymbol{x})\cdot\boldsymbol{T}_0(\boldsymbol{x}) = \boldsymbol{0} \tag{3}$$

for all $\boldsymbol{x}$ on ∂V.

2.1. *The equations of motion governing infinitesimal deformations*

We wish to investigate the possible free, infinitesimal, isentropic, elastic-gravitational deformations of this Earth model away from its equilibrium configuration. It is conventional to employ a mixed Lagrangian–Eulerian description of these deformations. Let $\boldsymbol{r}(\boldsymbol{x},t)$ denote the position vector in the inertial frame of reference of the material particle $\boldsymbol{x}$ at time t; the associated particle displacement $\boldsymbol{s}(\boldsymbol{x},t)$ is defined by

$$\boldsymbol{r}(\boldsymbol{x},t) = \boldsymbol{x}+\boldsymbol{s}(\boldsymbol{x},t). \tag{4}$$

Let $\mathbf{T}_{\mathrm{L}}(\boldsymbol{x},t)$ denote the non-symmetric Piola–Kirchoff stress tensor (Malvern 1969) at the material particle $\boldsymbol{x}$ at time t; the associated incremental Piola–Kirchoff stress tensor $\tilde{\boldsymbol{T}}(\boldsymbol{x},t)$ is defined by

$$\mathbf{T}_{\mathrm{L}}(\boldsymbol{x},t) = \boldsymbol{T}_0(\boldsymbol{x})+\tilde{\boldsymbol{T}}(\boldsymbol{x},t). \tag{5}$$

Let $\rho_{\mathrm{E}}(\boldsymbol{r},t)$ and $\phi_{\mathrm{E}}(\boldsymbol{r},t)$ denote, respectively, the Eulerian mass density and gravitational potential fields at the spatial point $\boldsymbol{r}(\boldsymbol{x},t)$ at time t. We define the incremental Eulerian density and gravitational potential fields $\rho_1(\boldsymbol{r},t)$ and $\phi_1(\boldsymbol{r},t)$ by

$$\rho_{\mathrm{E}}(\boldsymbol{r},t) = \rho_0(\boldsymbol{r})+\rho_1(\boldsymbol{r},t) \tag{6}$$

and

$$\phi_{\mathrm{E}}(\boldsymbol{r},t) = \phi_0(\boldsymbol{r})+\phi_1(\boldsymbol{r},t). \tag{7}$$

In the linearized analysis which follows, all of the incremental field variables $\boldsymbol{s}(\boldsymbol{x},t)$, $\tilde{\boldsymbol{T}}(\boldsymbol{x},t)$, $\rho_1(\boldsymbol{r},t)$ and $\phi_1(\boldsymbol{r},t)$ will be assumed to be infinitesimally small. We shall take advantage of the fact that $\rho_1(\boldsymbol{r},t) = \rho_1(\boldsymbol{x},t)$ and $\phi_1(\boldsymbol{r},t) = \phi_1(\boldsymbol{x},t)$, correct through terms of first order in the displacement $\boldsymbol{s}(\boldsymbol{x},t)$, to obtain a system of linearized field equations and associated boundary conditions over the underformed Earth model volume V and the underformed boundary ∂V.

A convenient method of generating both the field equations and the associated boundary conditions which govern $\boldsymbol{s}(\boldsymbol{x},t)$, $\tilde{\boldsymbol{T}}(\boldsymbol{x},t)$, $\rho_1(\boldsymbol{x},t)$ and $\phi_1(\boldsymbol{x},t)$ is to appeal to the principle of least action. The application of this principle requires the construction of a Lagrangian functional and, consequently, a consideration of the elastic and gravitational potential energies associated with an elastic-gravitational deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$. Expressions for these potential energies, as well as the corresponding Lagrangian, are given by Dahlen (1973); we summarize those results here.

We introduce a Cartesian axis system $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \hat{\boldsymbol{x}}_3$, having its origin at $\boldsymbol{O}$ and an arbitrary orientation within the volume V. We assume that the displacement gradient tensor $\boldsymbol{\nabla s}(\boldsymbol{x},t)$ is infinitesimal (in the sense that $|\partial_i s_j(\boldsymbol{x},t)| \ll 1$), so we may linearize the isentropic, perfectly elastic constitutive relation between the incremental Piola–Kirchoff stress tensor $\tilde{\boldsymbol{T}}(\boldsymbol{x},t)$ and the displacement gradient tensor $\boldsymbol{\nabla s}(\boldsymbol{x},t)$. Making use of the condition (3) on the free outer surface ∂V, the elastic potential energy functional $\mathfrak{U}[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ associated with a deformation $\boldsymbol{s}(\boldsymbol{x},t)$ takes the form

$$\mathfrak{U} = \int_V \mathrm{d}V\,[T^0_{ij}\tfrac{1}{2}(\partial_i s_j + \partial_j s_i)] + \frac{1}{2}\int_V \mathrm{d}V\,[\Lambda_{ijkl}\partial_i s_j \partial_k s_l]. \tag{8}$$

The $3^4 = 81$ coefficients $\Lambda_{ijkl}(\boldsymbol{x})$ are the Cartesian components of the fourth order isentropic elastic tensor $\boldsymbol{\Lambda}(\boldsymbol{x})$ which relates $\tilde{\boldsymbol{T}}(\boldsymbol{x},t)$ to $\boldsymbol{\nabla s}(\boldsymbol{x},t)$,

$$\tilde{T}_{ij} = \Lambda_{ijkl}\,\partial_k s_l. \tag{9}$$

The most general form of these components which is consistent with both the first and second laws of thermodynamics and the principle of material frame-indifference (Malvern 1969) is

$$\Lambda_{ijkl} = C_{ijkl} + \tfrac{1}{2}(T^0_{ij}\,\delta_{kl} + T^0_{kl}\,\delta_{ij} + T^0_{ik}\,\delta_{jl} - T^0_{jl}\,\delta_{ik} - T^0_{il}\,\delta_{jk} - T^0_{jk}\,\delta_{il}), \tag{10}$$

where the components $C_{ijkl}(\boldsymbol{x})$ possess the familiar symmetry relations of an elastic tensor,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{11}$$

(Equation (10) corrects a typographical sign error in equation (23) of Dahlen (1973).) The gravitational potential energy $\mathfrak{M}[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ associated with an elastic-gravitational deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ can be expressed as

$$\mathfrak{M} = \int_V \mathrm{d}V\,[\rho_0 s_j \partial_j \phi_0] + \frac{1}{2}\int_V \mathrm{d}V\,[2\rho_0 s_j \partial_j \phi_1 + \rho_0 s_i s_j \partial_i \partial_j \phi_0] + \frac{1}{2}\int_E \mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right], \tag{12}$$

correct through terms of second order in both $\boldsymbol{s}(\boldsymbol{x},t)$ and $\phi_1(\boldsymbol{x},t)$. Here E denotes all of space. The gravitational potential energy of the static equilibrium configuration has been adopted as the zero reference level. The kinetic energy $\mathfrak{T}[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ of a deformation $\boldsymbol{s}(\boldsymbol{x},t)$ with associated material particle velocity $\partial_t \boldsymbol{s}(\boldsymbol{x},t)$ is given by

$$\mathfrak{T} = \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0|\partial_t \boldsymbol{s}|^2]. \tag{13}$$

The Lagrangian functional $\mathfrak{L}[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ is defined by $\mathfrak{L} = \mathfrak{T} - (\mathfrak{U} + \mathfrak{M})$. Making use of the static equilibrium condition (2) and the free surface boundary condition (3) we can obtain

$$\mathfrak{L} = \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0|\partial_t \boldsymbol{s}|^2 - \Lambda_{ijkl}\partial_i s_j \partial_k s_l - 2\rho_0 s_j \partial_j \phi_1 - \rho_0 s_i s_j \partial_i \partial_j \phi_0] + \frac{1}{2}\int_E \mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right]. \tag{14}$$

Let $A[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ denote the action functional corresponding to a possible path of motion within some suitable configuration space of functions $\{\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)\}$ during some arbitrary time interval $t_1 \leqslant t \leqslant t_2$; the action is defined by

$$A[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)] = \int_{t_1}^{t_2} \mathrm{d}t\,\mathfrak{L}[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]. \tag{15}$$

The principle of least action asserts that $A[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ will be a stationary functional under independent small variations $\delta \boldsymbol{s}(\boldsymbol{x}, t)$ and $\delta\phi_1(\boldsymbol{x}, t)$ which vanish at times t_1 and t_2 if and only if $\boldsymbol{s}(\boldsymbol{x}, t)$ and $\phi_1(\boldsymbol{x}, t)$ constitute an admissible freely evolving elastic-gravitational deformation of the Earth. Application of this principle leads directly to the complete set of linearized field equations

$$\left.\begin{aligned} \rho_0 \partial_t^2 \boldsymbol{s} &= -\rho_0 \boldsymbol{\nabla}\phi_1 - \rho_0 \boldsymbol{s}\cdot\boldsymbol{\nabla\nabla}\phi_0 + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}}, \\ \nabla^2\phi_1 &= 4\pi G\rho_1, \\ \rho_1 &= -\boldsymbol{\nabla}\cdot(\rho_0 \boldsymbol{s}), \\ \tilde{\boldsymbol{T}} &= \boldsymbol{\Lambda} : \boldsymbol{\nabla s}, \end{aligned}\right\} \tag{16}$$

as well as to the associated linearized boundary conditions

$$\left.\begin{aligned} \hat{\boldsymbol{n}}\cdot\tilde{\boldsymbol{T}} &= \boldsymbol{0}, \\ [\phi_1]_-^+ &= 0, \\ [\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_1 + 4\pi G\rho_0 \hat{\boldsymbol{n}}\cdot\boldsymbol{s}]_-^+ &= 0, \end{aligned}\right\} \tag{17}$$

on the free surface ∂V. The notation $[\cdot]_-^+$ has here been used to denote the jump discontinuity in $[\cdot]$ upon going from outside V to inside V.

The first of the linearized equations (16) arises from the variation of $A[\boldsymbol{s}(\boldsymbol{x}, t), \phi_1(\boldsymbol{x}, t)]$ with respect to the displacement field $\boldsymbol{s}(\boldsymbol{x}, t)$; it expresses the local conservation of linear momentum. The second of the linearized equations (16) arises from the variation of $A[\boldsymbol{s}(\boldsymbol{x}, t), \phi_1(\boldsymbol{x}, t)]$ with respect to the incremental gravitational potential field $\phi_1(\boldsymbol{x}, t)$; it is an incremental version of Poisson's equation. The third and fourth of the linearized equations (16) simply serve to express the incremental density $\rho_1(\boldsymbol{x}, t)$ and the incremental Piola–Kirchoff stress $\tilde{\boldsymbol{T}}(\boldsymbol{x}, t)$ in terms of the displacement $\boldsymbol{s}(\boldsymbol{x}, t)$; the third is the linearized continuity equation and the fourth is the linearized isentropic elastic constitutive relation (9). The associated linearized boundary conditions (17) arise naturally and in a familiar fashion from the variational formulation as a result of the surface integral terms which are introduced by a number of integrations by parts, i.e. applications of Gauss's theorem. The first of the boundary conditions (17) expresses the fact that the surface ∂V is traction-free, while the second and third express the continuity of the incremental Eulerian gravitational potential and its normal derivative across the deformed outer surface. The particular form (12) of the gravitational potential energy functional $\mathfrak{M}[\boldsymbol{s}(\boldsymbol{x}, t), \phi_1(\boldsymbol{x}, t)]$ was chosen specifically to allow the action $A[\boldsymbol{s}(\boldsymbol{x}, t), \phi_1(\boldsymbol{x}, t)]$ to be varied independently with respect to the displacement $\boldsymbol{s}(\boldsymbol{x}, t)$ and the incremental gravitational potential $\phi_1(\boldsymbol{x}, t)$. The form (12) can best be motivated by the consideration of an alternative version of the variational principle, in which $\phi_1(\boldsymbol{x}, t)$ is treated as a functional of $\boldsymbol{s}(\boldsymbol{x}, t)$ and the incremental Poisson equation as well as the last two of the boundary conditions (17) are treated as constraints on the variation process, by means of the Lagrange multiplier method.

An important property of the free motion of any conservative system is the principle of conservation of energy. The appropriate form of this principle in the present case may be obtained by forming the dot product of the first of equations (16) with the particle velocity vector $\partial_t \boldsymbol{s}(\boldsymbol{x}, t)$, and integrating the result over V. Several applications of Gauss's theorem together with the boundary conditions (17) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int_V \mathrm{d}V[\rho_0|\partial_t \boldsymbol{s}|^2 + \Lambda_{ijkl}\partial_i s_j \partial_k s_l + 2\rho_0 s_j \partial_j \phi_1 + \rho_0 s_i s_j \partial_i \partial_j \phi_0] + \frac{1}{2}\int_{\mathrm{E}} \mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right]\right\} = 0. \tag{18}$$

The expression in braces in equation (18) is precisely the sum $\mathfrak{T} + \mathfrak{U} + \mathfrak{M}$ of the kinetic and the elastic gravitational potential energies associated with a freely evolving deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$. The principle of conservation of energy states that the total energy $\mathfrak{T}+\mathfrak{U}+\mathfrak{M}$ of any free deformation must remain constant for all times.

2.2. *Normal mode solutions and their properties*

We seek free solutions to the equations (16) and (17) in the form of a linear superposition of normal mode solutions $\boldsymbol{s}(\boldsymbol{x},t) = \boldsymbol{s}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}$, $\phi_1(\boldsymbol{x},t) = \phi_1(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}$. The various normal mode eigenfrequencies ω and the associated elastic-gravitational eigenfunctions $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ may be determined by solution of the time-independent elastic-gravitational field equations

$$\left.\begin{aligned} -\rho_0\omega^2\boldsymbol{s} &= -\rho_0\boldsymbol{\nabla}\phi_1-\rho_0\,\boldsymbol{s}\cdot\boldsymbol{\nabla\nabla}\phi_0+\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}},\\ \nabla^2\phi_1 &= 4\pi G\rho_1,\\ \rho_1 &= -\boldsymbol{\nabla}\cdot(\rho_0\,\boldsymbol{s}),\\ \tilde{\boldsymbol{T}} &= \boldsymbol{\Lambda}:\boldsymbol{\nabla s},\end{aligned}\right\}\tag{19}$$

subject to the boundary conditions

$$\left.\begin{aligned} \hat{\boldsymbol{n}}\cdot\tilde{\boldsymbol{T}} &= \boldsymbol{0},\\ [\phi_1]_-^+ &= 0,\\ [\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_1+4\pi G\rho_0\hat{\boldsymbol{n}}\cdot\boldsymbol{s}]_-^+ &= 0,\end{aligned}\right\}\tag{20}$$

on the free outer surface ∂V.

Equations (19) and (20) constitute a well-posed boundary value problem which admits eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$. We will consider explicitly the possibility that the eigenfrequencies ω might be complex numbers and that the associated eigenfunctions $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ might be complex fields. A number of useful and well-known properties of the complex normal mode eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ can be obtained by manipulation of the equations (19) and (20). Let $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ and $\{\omega', \boldsymbol{s}'(\boldsymbol{x}), \phi_1'(\boldsymbol{x})\}$ be any two different eigensolutions and define an inner product $(\boldsymbol{s}', \boldsymbol{s})$ between $\boldsymbol{s}'(\boldsymbol{x})$ and $\boldsymbol{s}(\boldsymbol{x})$ by

$$(\boldsymbol{s}', \boldsymbol{s}) = \int_V \mathrm{d}V\,[\rho_0\,\boldsymbol{s}'\cdot\boldsymbol{s}^*],\tag{21}$$

where the asterisk denotes complex conjugation. If we take the inner product (21) of $\boldsymbol{s}'(\boldsymbol{x})$ with the first of equations (19), and make use of the rest of equations (19) as well as the boundary conditions (20), we can arrive at the result

$$\omega^{*2}\mathscr{T}(\boldsymbol{s}', \boldsymbol{s})-\mathscr{E}(\boldsymbol{s}', \boldsymbol{s})-\mathscr{G}(\boldsymbol{s}', \boldsymbol{s}) = 0.\tag{22}$$

Equation (22) is a global relation between any two complex eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ and $\{\omega', \boldsymbol{s}'(\boldsymbol{x}), \phi_1'(\boldsymbol{x})\}$; it has been written in terms of the kinetic energy bilinear form $\mathscr{T}(\boldsymbol{s}', \boldsymbol{s})$, the elastic potential energy bilinear form $\mathscr{E}(\boldsymbol{s}', \boldsymbol{s})$, and the gravitational potential energy bilinear form $\mathscr{G}(\boldsymbol{s}', \boldsymbol{s})$. These are defined (following Dahlen (1973), but with a slight difference in notation) by

$$\mathscr{T}(\boldsymbol{s}', \boldsymbol{s}) = (\boldsymbol{s}', \boldsymbol{s}) = \int_V \mathrm{d}V\,[\rho_0\,\boldsymbol{s}'\cdot\boldsymbol{s}^*]\tag{23}$$

and

$$\left.\begin{aligned} \mathscr{E}(\boldsymbol{s}', \boldsymbol{s}) &= \int_V \mathrm{d}V\,[\Lambda_{ijkl}\partial_i s_j'\partial_k s_l^*],\\ \mathscr{G}(\boldsymbol{s}', \boldsymbol{s}) &= \int_V \mathrm{d}V\,[\rho_0 s_j'\partial_j\phi_1^*+\rho_0 s_j^*\partial_j\phi_1'+\rho_0 s_i' s_j^*\partial_i\partial_j\phi_0]+\int_{\mathrm{E}}\mathrm{d}V\left[\frac{1}{4\pi G}\partial_j\phi_1'\partial_j\phi_1^*\right].\end{aligned}\right\}\tag{24}$$

Reversal of the roles of the primed and unprimed eigensolutions leads immediately to the corresponding result

$$\omega'^{*2}\mathscr{T}(\boldsymbol{s}, \boldsymbol{s}')-\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}')-\mathscr{G}(\boldsymbol{s}, \boldsymbol{s}') = 0.\tag{25}$$

An important property of the three bilinear forms, (23) and (24), is that each is Hermitian symmetric, i.e. $\mathscr{T}(\boldsymbol{s}', \boldsymbol{s}) = \mathscr{T}^*(\boldsymbol{s}, \boldsymbol{s}')$, $\mathscr{E}(\boldsymbol{s}', \boldsymbol{s}) = \mathscr{E}^*(\boldsymbol{s}, \boldsymbol{s}')$ and $\mathscr{G}(\boldsymbol{s}', \boldsymbol{s}) = \mathscr{G}^*(\boldsymbol{s}, \boldsymbol{s}')$. This property is evident from inspection of the defining relations (23) and (24), taking into account the symmetry relation $\Lambda_{ijkl} = \Lambda_{klij}$. If we allow the primed and unprimed eigensolutions to be the same in either of equations (22) or (25), we obtain

$$\omega^{*2}\mathscr{T}(\boldsymbol{s}, \boldsymbol{s}) - \mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) - \mathscr{G}(\boldsymbol{s}, \boldsymbol{s}) = 0. \tag{26}$$

Hermitian symmetry implies that each of $\mathscr{T}(\boldsymbol{s}, \boldsymbol{s})$, $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s})$ and $\mathscr{G}(\boldsymbol{s}, \boldsymbol{s})$ must be purely real; complex conjugation of (26) thus leads to

$$\omega^{2}\mathscr{T}(\boldsymbol{s}, \boldsymbol{s}) - \mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) - \mathscr{G}(\boldsymbol{s}, \boldsymbol{s}) = 0. \tag{27}$$

Equation (27) is an important global relationship which must be satisfied by every eigensolution $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$; it has numerous applications and consequences. One immediate consequence is the fact that every squared eigenfrequency ω^2 of a non-rotating Earth model must be real. The reality of both ω^2 and ω'^2 in equations (22) and (25) leads directly to another well-known result, namely

$$\mathscr{T}(\boldsymbol{s}', \boldsymbol{s}) = (\boldsymbol{s}', \boldsymbol{s}) = 0 \quad \text{if} \quad \omega'^2 - \omega^2 \neq 0. \tag{28}$$

Complex displacement eigenfunctions associated with distinct squared eigenfrequencies are orthogonal in the sense of the inner product (21).

Equation (27) may also be used to investigate the mechanical stability of a non-rotating model of the Earth. Any non-rotating equilibrium configuration will be completely stable against infinitesimal elastic-gravitational disturbances if all of its squared elastic-gravitational eigenfrequencies ω^2 are not only real, but non-negative (i.e. provided that none of the eigenfrequencies ω are purely imaginary). Since $\mathscr{T}(\boldsymbol{s}, \boldsymbol{s}) = (\boldsymbol{s}, \boldsymbol{s})$ is positive definite, equation (27) implies that a necessary and sufficient condition for complete stability is that $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) + \mathscr{G}(\boldsymbol{s}, \boldsymbol{s})$ be non-negative for all admissible static infinitesimal deformations $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$. An *admissible* static deformation is in this instance defined to be one for which the incremental gravitational potential $\phi_1(\boldsymbol{x})$ is related to the displacement $\boldsymbol{s}(\boldsymbol{x})$ through the incremental Poisson equation and the last two of the boundary conditions (20). The quantity $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) + \mathscr{G}(\boldsymbol{s}, \boldsymbol{s})$ is precisely twice the sum $\mathfrak{U} + \mathfrak{M}$ of the elastic and gravitational potential energies associated with the purely real static deformation $\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})] = \frac{1}{2}[\boldsymbol{s}(\boldsymbol{x}) + \boldsymbol{s}^*(\boldsymbol{x})]$, $\mathrm{Re}[\phi_1(\boldsymbol{x})] = \frac{1}{2}[\phi_1(\boldsymbol{x}) + \phi_1^*(\boldsymbol{x})]$. We may thus assert that a non-rotating Earth model will be a completely stable elastic-gravitational equilibrium configuration if and only if it is not possible to decrease its net elastic-gravitational potential energy by means of a small deformation $\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]$, $\mathrm{Re}[\phi_1(\boldsymbol{x})]$, i.e. if and only if its elastic-gravitational potential energy is a local minimum in the equilibrium configuration. This is a familiar and physically plausible stability criterion; we will henceforth assume that the system under study is completely stable in this sense. This is almost surely a reasonable assumption for an everywhere solid, but otherwise 'typical' Earth model (although non-rotating Earth models which have an extremely high ratio of the mass density to the incompressibility may be gravitationally unstable (Love 1911)). The stability of non-rotating Earth models with a fluid outer core and/or oceans is known to depend rather critically on the details of the density stratification in the fluid portions (see, for example, Eckart 1960; Dahlen 1974).

The form of the equations governing the normal mode boundary value problem (19) and (20) can be shown to endow the class of all eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ with a certain minimum or primitive algebraic structure, independent of any possible symmetries in the Earth model. We will examine this structure in some detail, since we will later exploit it to establish a one-to-one

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

correspondence between the normal modes of oscillation of a non-rotating Earth model and those of any related rotating Earth model. Consider the group of four tentative eigensolutions: $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$, $\{-\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$, $\{\omega, \boldsymbol{s}^*(\boldsymbol{x}), \phi_1^*(\boldsymbol{x})\}$ and $\{-\omega, \boldsymbol{s}^*(\boldsymbol{x}), \phi_1^*(\boldsymbol{x})\}$. Since the equations (19) and (20) depend only on the squares of the eigenfrequencies ω^2, and since every ω^2 is necessarily real, it is readily shown that if any member of this quartet is an eigensolution, then every member of the quartet is an eigensolution. Associated with every such quartet of eigensolutions, there is a two-dimensional complex eigenspace of displacement eigenfunctions; this is the space of complex vector-valued functions over V spanned by the linearly independent complex conjugate pair $\boldsymbol{s}(\boldsymbol{x})$ and $\boldsymbol{s}^*(\boldsymbol{x})$ (with an exception noted in the next paragraph). We consider every such two-dimensional complex eigenspace to be associated with four eigenfrequencies, i.e. a pair of equal eigenfrequencies at each of ω and $-\omega$.

There may, in general, be eigenfunctions of a non-rotating Earth model which satisfy $\boldsymbol{s}(\boldsymbol{x}) = \alpha\boldsymbol{s}^*(\boldsymbol{x})$, $\phi_1(\boldsymbol{x}) = \alpha\phi_1^*(\boldsymbol{x})$, where α is some complex constant. In this case, $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ and $\boldsymbol{s}^*(\boldsymbol{x})$, $\phi_1^*(\boldsymbol{x})$ are not linearly independent, and only two members of the associated quartet, say $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ and $\{-\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$, are distinct. The complex eigenspace of displacement eigenfunctions which is associated with any such degenerate quartet has dimension one, and the associated eigenfrequencies are two in number, ω and $-\omega$.

Any geometrical symmetry of the Earth model will, in general, lead to the phenomenon of eigenfrequency degeneracy (there might also be accidental eigenfrequency degeneracy which does not arise from symmetries in the Earth model). Every degenerate eigenfrequency ω will have an associated complex eigenspace of displacement eigenfunctions of some dimension $n > 1$. Every such n-dimensional degenerate complex eigenspace can always be decomposed into disjoint one- and two-dimensional primitive subspaces of the type discussed above. We therefore may associate with every n-dimensional degenerate complex eigenspace $2n$ eigenfrequencies, i.e. an n-tuple of eigenfrequencies at each of ω and $-\omega$.

This profligate multiplicity of normal mode eigenfrequencies and eigenfunctions is introduced here primarily to facilitate comparison with the case of a rotating Earth model. Rotation will in general lead to splitting of degenerate eigenfrequencies. If the eigenfrequencies of a non-rotating Earth model are counted in the manner described above, there will always be a one-to-one correspondence between the eigenfrequencies of any non-rotating Earth model and those of its rotating counterpart.

There are two linearly independent real normal modes of oscillation $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$ associated with any quartet of eigensolutions, and consequently with any two-dimensional primitive complex eigenspace. These can be written in the form

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \alpha\boldsymbol{s}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}+\alpha^*\boldsymbol{s}^*(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t} = 2\,\mathrm{Re}[\alpha\boldsymbol{s}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}],\\ \phi_1(\boldsymbol{x},t) &= \alpha\phi_1(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}+\alpha^*\phi_1^*(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t} = 2\,\mathrm{Re}[\alpha\phi_1(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}] \end{aligned}\right\} \tag{29}$$

and

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \beta\boldsymbol{s}(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}+\beta^*\boldsymbol{s}^*(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t} = 2\,\mathrm{Re}[\beta\boldsymbol{s}(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}],\\ \phi_1(\boldsymbol{x},t) &= \beta\phi_1(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}+\beta^*\phi_1^*(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t} = 2\,\mathrm{Re}[\beta\phi_1(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}], \end{aligned}\right\} \tag{30}$$

where α and β are arbitrary complex constants. The expressions (29) and (30) represent, in a generalized sense, travelling wave solutions $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$ to the elastic-gravitational field equations (16) and (17). The two travelling waves (29) and (30) have the same generalized phase speed, but travel in opposite directions. This circumstance allows the alternative construction of standing wave solutions $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$. Two linearly independent standing wave representations which may be composed out of the same quartet of eigensolutions are

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= [\boldsymbol{s}(\boldsymbol{x})+\boldsymbol{s}^*(\boldsymbol{x})]\,[\mu\,\mathrm{e}^{\mathrm{i}\omega t}+\mu^*\,\mathrm{e}^{-\mathrm{i}\omega t}] = 4\,\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]\,\mathrm{Re}[\mu\,\mathrm{e}^{\mathrm{i}\omega t}],\\ \phi_1(\boldsymbol{x},t) &= [\phi_1(\boldsymbol{x})+\phi_1^*(\boldsymbol{x})]\,[\mu\,\mathrm{e}^{\mathrm{i}\omega t}+\mu^*\,\mathrm{e}^{-\mathrm{i}\omega t}] = 4\,\mathrm{Re}[\phi_1(\boldsymbol{x})]\,\mathrm{Re}[\mu\,\mathrm{e}^{\mathrm{i}\omega t}]\end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= [\boldsymbol{s}(\boldsymbol{x})-\boldsymbol{s}^*(\boldsymbol{x})]\,[\nu\,\mathrm{e}^{\mathrm{i}\omega t}-\nu^*\,\mathrm{e}^{-\mathrm{i}\omega t}] = -4\,\mathrm{Im}[\boldsymbol{s}(\boldsymbol{x})]\,\mathrm{Im}[\nu\,\mathrm{e}^{\mathrm{i}\omega t}],\\ \phi_1(\boldsymbol{x},t) &= [\phi_1(\boldsymbol{x})-\phi_1^*(\boldsymbol{x})]\,[\nu\,\mathrm{e}^{\mathrm{i}\omega t}-\nu^*\,\mathrm{e}^{-\mathrm{i}\omega t}] = -4\,\mathrm{Im}[\phi_1(\boldsymbol{x})]\,\mathrm{Im}[\nu\,\mathrm{e}^{\mathrm{i}\omega t}],\end{aligned}\right\} \tag{32}$$

where μ and ν are arbitrary complex constants.

There is only one real normal mode of oscillation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ associated with any degenerate quartet of eigensolutions which has an associated one-dimensional primitive complex eigenspace. This necessarily has the form of a single standing wave

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]\,[\sigma\,\mathrm{e}^{\mathrm{i}\omega t}+\sigma^*\,\mathrm{e}^{-\mathrm{i}\omega t}] = 2\,\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]\,\mathrm{Re}[\sigma\,\mathrm{e}^{\mathrm{i}\omega t}],\\ \phi_1(\boldsymbol{x},t) &= \mathrm{Re}[\phi_1(\boldsymbol{x})]\,[\sigma\,\mathrm{e}^{\mathrm{i}\omega t}+\sigma^*\,\mathrm{e}^{-\mathrm{i}\omega t}] = 2\,\mathrm{Re}[\phi_1(\boldsymbol{x})]\,\mathrm{Re}[\sigma\,\mathrm{e}^{\mathrm{i}\omega t}],\end{aligned}\right\} \tag{33}$$

where σ is an arbitrary complex constant.

An important difference between the travelling wave representations (29) and (30) and the standing wave representations (31), (32) and (33) of the various normal modes of oscillation is that the particle motion $\boldsymbol{s}(\boldsymbol{x},t)$ and the associated incremental gravitational potential $\phi_1(\boldsymbol{x},t)$ of the latter are everywhere in V either in phase, or exactly out of phase by π radians. Whenever the standing wave representation is employed for all of the normal modes of oscillation of a non-rotating Earth model, it is generally convenient to simply dispense with the possibility of complex eigenfunctions and to work entirely with real ones. We can replace every primitive two-dimensional complex eigenspace spanned by $\boldsymbol{s}(x)$ and $\boldsymbol{s}^*(\boldsymbol{x})$ by the two-dimensional real eigenspace spanned by $\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]$ and $\mathrm{Im}[\boldsymbol{s}(\boldsymbol{x})]$, and we can replace every primitive one-dimensional complex eigenspace spanned by $\boldsymbol{s}(\boldsymbol{x})$ by the one-dimensional real eigenspace spanned by $\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]$. We have admitted the possibility of complex eigenfunctions and pointed out the associated travelling wave representations (29) and (30) of the normal modes of oscillation because these will arise naturally in the rotating case.

Every linearly independent normal mode of oscillation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ of the form (29), (30), (31), (32) or (33) requires, as part of its specification, the assignment of some complex constant. Each such constant may be regarded as determing the amplitude and the absolute phase of a real normal mode of oscillation. A well-posed initial value problem for the free elastic-gravitational deformation of a non-rotating Earth model requires the specification, at some initial instant $t = 0$, of both the initial displacement field $\boldsymbol{s}(\boldsymbol{x},0)$ and the initial particle velocity $\partial_t\,\boldsymbol{s}(\boldsymbol{x},0)$ (the initial values $\phi_1(\boldsymbol{x},0)$ and $\partial_t\,\phi_1(\boldsymbol{x},0)$ are then uniquely defined by the incremental Poisson equation and the last two of the boundary conditions (17)). This Cauchy initial value prescription is both necessary and sufficient for the determination of the resulting amplitude and absolute phase of each real normal mode of oscillation. We shall associate a single degree of freedom of the elastic-gravitational deformation of a non-rotating Earth model with each linearly independent normal mode of oscillation. We thus associate a single degree of freedom with each one-dimensional primitive complex eigenspace, and two degrees of freedom with each two-dimensional primitive complex eigenspace. More generally, the number of degrees of freedom associated with any complex eigenspace is equal to the dimension of that eigenspace.

A general property of every normal mode of oscillation of a non-rotating Earth model is the principle of equipartition of energy. Over any integral number of cycles, the averaged kinetic energy $\mathfrak{T}[s(x,t), \phi_1(x,t)]$ associated with any of the normal modes of oscillation (29), (30), (31), (32) or (33) is equal to the averaged elastic-gravitational potential energy

$$\mathfrak{U}[s(x,t), \phi_1(x,t)] + \mathfrak{M}[s(x,t), \phi_1(x,t)].$$

This equality is in fact expressed by equation (27), and for that reason, we will refer to equation (27) as the *equipartition equation*.

It is sometimes useful to regard the equipartition equation in an alternative role as a quadratic equation in ω, the coefficients of which are uniquely defined by a particular complex eigenfunction $s(x)$, $\phi_1(x)$. The roots of any quadratic equation of the form (27) are necessarily of the form $\pm\omega$. The content of the equation is that if $s(x)$, $\phi_1(x)$ is any complex elastic-gravitational eigenfunction, then $\pm\omega$ are the two eigenfrequencies associated with that eigenfunction if and only if $\pm\omega$ are the roots of the quadratic equipartition equation defined by $s(x)$, $\phi_1(x)$. The various members of any one-dimensional primitive complex eigenspace must all have the same associated eigenfrequencies $\pm\omega$, and they in fact all give rise to the same quadratic equation, apart from a real multiplicative constant. The same remark applies to the various members of any two-dimensional primitive complex eigenspace as well; the respective quadratic equations defined by any complex conjugate pair of eigenfunctions $s(x)$, $\phi_1(x)$ and $s^*(x)$, $\phi_1^*(x)$ are identical, because of the relations

$$\mathscr{T}(s^*, s^*) = \mathscr{T}(s, s), \quad \mathscr{E}(s^*, s^*) = \mathscr{E}(s, s) \quad \text{and} \quad \mathscr{G}(s^*, s^*) = \mathscr{G}(s, s).$$

One of the most useful properties of the eigensolutions of a conservative system is the variational principle first enunciated by Rayleigh (1894). If we regard the left-hand side of the equipartition equation (27) as defining a functional $\omega^2\mathscr{T}(s,s) - \mathscr{E}(s,s) - \mathscr{G}(s,s)$ of the elastic-gravitational disturbance $s(x)$, $\phi_1(x)$, Rayleigh's principle asserts that this functional is stationary under independent small variations $\delta s(x)$, $\delta\phi_1(x)$ if and only if $s(x)$, $\phi_1(x)$ is an elastic-gravitational eigenfunction with squared eigenfrequency ω^2. This principle can be shown to be an immediate consequence of the Hermitian symmetry of the bilinear forms $\mathscr{T}(s', s)$, $\mathscr{E}(s', s)$ and $\mathscr{G}(s', s)$, or it may alternatively be shown to follow directly from the more fundamental principle of least action. Extensive use has been made of Rayleigh's principle to determine the influence of small perturbations in the Earth's physical properties on its normal mode eigenfrequencies; the most complete such study to date is that of Luh (1974).

In any investigation of the excitation of the normal modes of oscillation of a non-rotating Earth model, a unique role is played by those normal modes which are associated with the eigenfrequency $\omega = 0$. Since $\mathscr{T}(s,s) = (s,s)$ is positive definite, the equipartition equation (27) implies that the zero-frequency normal mode eigenfunctions $s(x)$, $\phi_1(x)$ must satisfy $\mathscr{E}(s,s) + \mathscr{G}(s,s) = 0$. The eigenspace of all zero-frequency eigenfunctions consists precisely of all those complex elastic-gravitational deformations $s(x)$, $\phi_1(x)$ which do not alter the net elastic-gravitational potential energy of the equilibrium configuration; we might, in a sense, regard the zero-frequency eigenspace as the symmetry or invariance group of the elastic-gravitational potential energy functional $\mathscr{E}(s,s) + \mathscr{G}(s,s)$. The zero-frequency eigenspace of any non-rotating Earth model can always be decomposed into disjoint one- and two-dimensional primitive subspaces, just as can any other degenerate eigenspace. Because all the eigenfrequencies associated with these primitive subspaces vanish, we cannot usefully express the associated real normal modes of oscillation $s(x,t)$, $\phi_1(x,t)$ in the form of travelling or standing waves. We must instead associate with every two-dimensional primitive zero-frequency eigenspace spanned by $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ and $\boldsymbol{s}^*(\boldsymbol{x})$, $\phi_1^*(\boldsymbol{x})$ the real secular motions

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})][A+Bt], \\ \phi_1(\boldsymbol{x},t) &= \mathrm{Re}[\phi_1(\boldsymbol{x})][A+Bt] \end{aligned}\right\} \tag{34}$$

and

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \mathrm{Im}[\boldsymbol{s}(\boldsymbol{x})][C+Dt], \\ \phi_1(\boldsymbol{x},t) &= \mathrm{Im}[\phi_1(\boldsymbol{x})][C+Dt], \end{aligned}\right\} \tag{35}$$

where A, B, C and D are arbitrary real constants; we must likewise associate with every one-dimensional primitive zero-frequency eigenspace spanned by $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ the real secular motion

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})][E+Ft], \\ \phi_1(\boldsymbol{x},t) &= \mathrm{Re}[\phi_1(\boldsymbol{x})][E+Ft], \end{aligned}\right\} \tag{36}$$

where E and F are arbitrary real constants. The constants A, B, C, D, E and F required in the specification of the modes of motion (34), (35) and (36) can be uniquely determined by a Cauchy initial value prescription. The zero-frequency normal modes of a non-rotating Earth model might also be called the secular modes, because of the property that the associated real motions $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ contain a secular term which may grow linearly with time.

It is clear that every Earth model must have at least those zero-frequency normal modes that correspond to rigid body translations and rotations of that Earth model in an inertial frame of reference. It we restrict consideration to purely real eigenfunctions, we can express the eigenfunctions $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ which correspond to the rigid body translations and rotations in the form

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x}) &= \boldsymbol{R}+\boldsymbol{Q}\cdot\boldsymbol{x}, \\ \phi_1(\boldsymbol{x}) &= -\boldsymbol{s}(\boldsymbol{x})\cdot\boldsymbol{\nabla}\phi_0(\boldsymbol{x}), \end{aligned}\right\} \tag{37}$$

where $\boldsymbol{R}$ is an arbitrary constant vector and $\boldsymbol{Q}$ is an arbitrary constant proper orthogonal tensor. The relations (37) describe a six-dimensional space of real eigenfunctions $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ associated with the 6 degrees of freedom of rigid body motions; associated with this space are 12 eigenfrequencies, all having the value $\omega = 0$.

It is of some interest to determine what becomes of the six-dimensional rigid body eigenspace when the Earth model is allowed to rotate. It will be useful for that purpose not only to admit the possibility of complex rigid body eigenfunctions in the non-rotating case, but also to establish a useful decomposition of the six-dimensional complex rigid body eigenspace into disjoint one- and two-dimensional primitive subspaces. Let $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, be the coordinate vectors of the right-handed spherical polar coordinate system which is related to the Cartesian coordinate vectors $\hat{\boldsymbol{x}}_1$, $\hat{\boldsymbol{x}}_2$, $\hat{\boldsymbol{x}}_3$ in the usual way, and let $Y_l^m(\theta,\phi)$ be the fully normalized complex surface spherical harmonics defined with respect to that polar coordinate system (to be specific, we shall suppose them to follow the phase and normalization conventions of Edmonds (1960)). Let

$$\boldsymbol{\nabla}_1 = \hat{\boldsymbol{\theta}}\,\partial_\theta + \hat{\boldsymbol{\phi}}(\sin\theta)^{-1}\,\partial_\phi$$

be the surface gradient operator. If σ is an arbitrary complex constant, spheroidal vector fields of the form

$$\boldsymbol{s}(\boldsymbol{x}) = \sigma[\hat{\boldsymbol{r}}Y_1^0(\theta,\phi) + \boldsymbol{\nabla}_1 Y_1^0(\theta,\phi)] \tag{38}$$

comprise a one-dimensional primitive complex eigenspace of displacement eigenfunctions corresponding to the rigid body translations along the axis $\hat{\boldsymbol{x}}_3$. If α and β are arbitrary complex constants, spheroidal vector fields of the form

$$\begin{aligned} \boldsymbol{s}(\boldsymbol{x}) &= \alpha[\hat{\boldsymbol{r}}Y_1^1(\theta,\phi) + \boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)] + \beta[\hat{\boldsymbol{r}}Y_1^1(\theta,\phi) + \boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)]^* \\ &= \alpha[\hat{\boldsymbol{r}}Y_1^1(\theta,\phi) + \boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)] - \beta[\hat{\boldsymbol{r}}Y_1^{-1}(\theta,\phi) + \boldsymbol{\nabla}_1 Y_1^{-1}(\theta,\phi)] \end{aligned} \tag{39}$$

comprise a two-dimensional primitive complex eigenspace of displacement eigenfunctions corresponding to the rigid body translations orthogonal to the axis $\hat{\boldsymbol{x}}_3$. Similarly, toroidal vector fields of the form

$$\boldsymbol{s}(\boldsymbol{x}) = \sigma[-\boldsymbol{r}\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^0(\theta, \phi)] \tag{40}$$

comprise a one-dimensional primitive complex eigenspace corresponding to rigid body rotations about the axis $\hat{\boldsymbol{x}}_3$, and those of the form

$$\begin{aligned}\boldsymbol{s}(\boldsymbol{x}) &= \alpha[-r\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^1(\theta, \phi)] + \beta[-r\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^1(\theta, \phi)]^* \\ &= \alpha[-r\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^1(\theta, \phi)] - \beta[-r\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^{-1}(\theta, \phi)]\end{aligned} \tag{41}$$

comprise a two-dimensional primitive complex eigenspace corresponding to the rigid body rotations about axes orthogonal to $\hat{\boldsymbol{x}}_3$. The six-dimensional complex rigid body eigenspace is the direct sum of all spheroidal and toroidal vector fields of the form (38), (39), (40) and (41). Each of the one- and two-dimensional primitive complex eigenspaces is associated with, respectively, two and four eigenfrequencies, all having the value $\omega = 0$.

In the case of any non-rotating Earth model which is everywhere solid, and which is not on the verge of gravitational instability, we may reasonably assume that the rigid body modes will be the only modes with zero associated eigenfrequency. Earth models which have a fluid outer core and/or oceans will, on the other hand, in general have an infinite number of additional zero-frequency normal modes, corresponding to the fact that any deformation consisting entirely of dilationless fluid flow along equipotential surfaces will not alter the elastic-gravitational potential energy. The phenomena which arise in the rotating case as a result of this infinite class of zero-frequency modes in the non-rotating limit are numerous and profound, but they will not be addressed in this paper. We will henceforth assume that we are dealing with an Earth model which, in the non-rotating limit, has no zero-frequency modes other than those corresponding to rigid body motion. We intend here to focus attention on certain aspects of the influence of rotation which are more general than, and distinct from, the complications introduced by the presence of fluid portions in the Earth model. In so doing, we are simply taking the point of view that theoretical models of the behaviour of the rotating Earth must be constructed by accounting for one additional complication at a time.

2.3. *Normal mode excitation by a transient source*

We consider now the excitation of the elastic-gravitational free oscillations of a non-rotating Earth model by the action of an imposed body force distribution. We suppose that prior to time $t = 0$, the Earth model is at rest in the equilibrium configuration, so that $\boldsymbol{s}(\boldsymbol{x}, 0) = \boldsymbol{0}$, $\partial_t \boldsymbol{s}(\boldsymbol{x}, 0) = \boldsymbol{0}$, $\phi_1(\boldsymbol{x}, 0) = \boldsymbol{0}$ and $\partial_t \phi_1(\boldsymbol{x}, 0) = \boldsymbol{0}$; the more general case of arbitrary initial conditions can be straightforwardly handled, and we omit it solely for brevity. Beginning at $t = 0$, a transient body force distribution begins to act. We let $\boldsymbol{f}(\boldsymbol{x}, t)$ denote the Lagrangian description of the imposed body force density, measured per unit mass. We assume that the temporal variation of $\boldsymbol{f}(\boldsymbol{x}, t)$ has a finite duration T, and that at time T, $\boldsymbol{f}(\boldsymbol{x}, t)$ assumes a final value $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ which is possibly different from zero; thus $\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{0}$ for $t < 0$ and $\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ for $t \geqslant T$. We wish to determine the elastic-gravitational response of the Earth model to this imposed force. We will consider explicitly only the displacement response $\boldsymbol{s}(\boldsymbol{x}, t)$; the associated gravitational perturbation $\phi_1(\boldsymbol{x}, t)$ can always be determined from $\boldsymbol{s}(\boldsymbol{x}, t)$ by solving the incremental Poisson equation subject to the last two of the boundary conditions (17). In any physically realizable Earth model, the free oscillations excited by a transient body force will, in finite time, decay to an arbitrarily small amplitude as a result of imperfections of elasticity. We seek, therefore, to determine not only the dynamic response $\boldsymbol{s}(\boldsymbol{x}, t)$ for times relatively soon after $t = 0$, but also the final static response $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$ after all the free oscillations have been caused to decay by the introduction of a small amount of dissipation. The treatment follows that of Gilbert (1971).

It is clear that the presence of zero-frequency or secular normal modes can pose a problem to any excitation calculation. Any secular mode has the property that, if excited, it may eventually lead to an arbitrarily large deviation away from the equilibrium configuration, no matter how infinitesimal is its amplitude of excitation. Arbitrarily large displacements $\boldsymbol{s}(\boldsymbol{x}, t)$ are not formally compatible with the linearized theory used here. If the only secular modes are those arising from the six rigid body degrees of freedom, we may alleviate this difficulty by requiring that the imposed body force distribution $\boldsymbol{f}(\boldsymbol{x}, t)$ exert neither a net force nor a net torque on the Earth model, i.e.

$$\left.\begin{aligned} \int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, t)] &= \boldsymbol{0}, \\ \int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x})\,\boldsymbol{x}\times\boldsymbol{f}(\boldsymbol{x}, t)] &= \boldsymbol{0}. \end{aligned}\right\} \tag{42}$$

The constraints (42) must be satisfied for all times t. They will always be satisfied whenever $\boldsymbol{f}(\boldsymbol{x}, t)$ represents a physically realizable process which is wholly internal to the Earth; in particular, they will always be satisfied whenever $\boldsymbol{f}(\boldsymbol{x}, t)$ is the body force distribution equivalent to some postulated earthquake source mechanism.

The dynamical equations which govern the elastic-gravitational response $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$ to the imposed body force distribution $\boldsymbol{f}(\boldsymbol{x}, t)$, for $t \geqslant 0$, are

$$\left.\begin{aligned} \rho_0\,\partial_t^2 \boldsymbol{s} &= -\rho_0 \boldsymbol{\nabla}\phi_1 - \rho_0\,\boldsymbol{s}\cdot\boldsymbol{\nabla\nabla}\phi_0 + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}} + \rho_0 \boldsymbol{f}, \\ \nabla^2\phi_1 &= 4\pi G\rho_1, \\ \rho_1 &= -\boldsymbol{\nabla}\cdot(\rho_0\,\boldsymbol{s}), \\ \tilde{\boldsymbol{T}} &= \boldsymbol{\Lambda} : \boldsymbol{\nabla}\boldsymbol{s}, \end{aligned}\right\} \tag{43}$$

together with the dynamical free surface boundary conditions (17). We define the Laplace transform, $\bar{\boldsymbol{f}}(\boldsymbol{x}, p)$, of a function $\boldsymbol{f}(\boldsymbol{x}, t)$ by

$$\bar{\boldsymbol{f}}(\boldsymbol{x}, p) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt}\boldsymbol{f}(\boldsymbol{x}, t) \quad (\operatorname{Re} p > 0), \tag{44}$$

and we apply this transform to the equations (43), implicitly assuming the integral (44) to exist wherever necessary. We make the fundamental assumption that the complex normal mode displacement eigenfunctions $\boldsymbol{s}(\boldsymbol{x})$ of the Earth model form a complete linear space of complex vector-valued functions over V. Let $\boldsymbol{s}_n(\boldsymbol{x})$, $1 \leqslant n \leqslant \infty$, be a set of basis eigenfunctions spanning this linear space, and orthonormal in the sense of the inner product (27), i.e.

$$(\boldsymbol{s}_n, \boldsymbol{s}_m) = \mathscr{T}(\boldsymbol{s}_n, \boldsymbol{s}_m) = \delta_{nm}. \tag{45}$$

We are free to choose a set of orthonormal basis eigenfunctions because of the result (28) that displacement eigenfunctions with distinct squared eigenfrequencies ω^2 are necessarily orthogonal. It is also always possible, and generally convenient, to choose orthonormal basis eigenfunctions from within any given degenerate eigenspace in such a way that every one-dimensional primitive complex subspace contributes one purely real basis element, and every two-dimensional primitive complex subspace contributes two basis elements which are a complex conjugate pair. We represent all of $\bar{\boldsymbol{s}}(\boldsymbol{x},p)$, $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$, $\bar{\boldsymbol{f}}(\boldsymbol{x},p)$ and $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ as a linear superposition of the basis eigenfunctions $\boldsymbol{s}_n(\boldsymbol{x})$,

$$\left.\begin{aligned} \bar{\boldsymbol{s}}(\boldsymbol{x},p) &= \sum_{n=1}^{\infty} a_n(p)\, \boldsymbol{s}_n(\boldsymbol{x}), \\ \boldsymbol{s}_{\text{final}}(\boldsymbol{x}) &= \sum_{n=1}^{\infty} a_n^{\text{final}}\, \boldsymbol{s}_n(\boldsymbol{x}), \\ \bar{\boldsymbol{f}}(\boldsymbol{x},p) &= \sum_{n=1}^{\infty} f_n(p)\, \boldsymbol{s}_n(\boldsymbol{x}), \\ \boldsymbol{f}_{\text{final}}(\boldsymbol{x}) &= \sum_{n=1}^{\infty} f_n^{\text{final}}\, \boldsymbol{s}_n(\boldsymbol{x}). \end{aligned}\right\} \tag{46}$$

The orthonormality (45) of the basis eigenfunctions leads immediately to

$$\left.\begin{aligned} a_n(p) &= (\bar{\boldsymbol{s}}, \boldsymbol{s}_n) = \int_V \mathrm{d}V[\rho_0(\boldsymbol{x})\, \bar{\boldsymbol{s}}(\boldsymbol{x},p)\cdot \boldsymbol{s}_n^*(\boldsymbol{x})], \\ a_n^{\text{final}} &= (\boldsymbol{s}_{\text{final}}, \boldsymbol{s}_n) = \int_V \mathrm{d}V[\rho_0(\boldsymbol{x})\, \boldsymbol{s}_{\text{final}}(\boldsymbol{x})\cdot \boldsymbol{s}_n^*(\boldsymbol{x})], \\ f_n(p) &= (\bar{\boldsymbol{f}}, \boldsymbol{s}_n) = \int_V \mathrm{d}V[\rho_0(\boldsymbol{x})\bar{\boldsymbol{f}}(\boldsymbol{x},p)\cdot \boldsymbol{s}_n^*(\boldsymbol{x})], \\ f_n^{\text{final}} &= (\boldsymbol{f}_{\text{final}}, \boldsymbol{s}_n) = \int_V \mathrm{d}V[\rho_0(\boldsymbol{x})\boldsymbol{f}_{\text{final}}(\boldsymbol{x})\cdot \boldsymbol{s}_n^*(\boldsymbol{x})]. \end{aligned}\right\} \tag{47}$$

If we insert the expansions (46) into the Laplace transformed version of the dynamical equations (43), and take the inner product of the resultant equations with some particular basis eigenfunction, say $\boldsymbol{s}_m(\boldsymbol{x})$, we obtain, after some applications of Gauss's theorem together with the normal mode boundary conditions (20),

$$p^2 \sum_{n=1}^{\infty} a_n(\boldsymbol{s}_n, \boldsymbol{s}_m) + \sum_{n=1}^{\infty} a_n[\mathscr{E}(\boldsymbol{s}_n, \boldsymbol{s}_m) + \mathscr{G}(\boldsymbol{s}_n, \boldsymbol{s}_m)] = \sum_{n=1}^{\infty} f_n(\boldsymbol{s}_n, \boldsymbol{s}_m). \tag{48}$$

We can now make use of either equation (22) or (25) and of the orthonormality (45) of the basis eigenfunctions to obtain a separate, decoupled equation for each of the $a_m(p)$, namely

$$a_m(p) = f_m(p)/(p^2+\omega_m^2). \tag{49}$$

It is evident that the conditions (42) on $\boldsymbol{f}(\boldsymbol{x},t)$ lead directly to the result that $a_m(p)$ vanishes when $\boldsymbol{s}_m(\boldsymbol{x})$ is any one of the zero-frequency rigid body eigenfunctions. This is simply the well-known result that force distributions which exert no net force and no net torque upon the Earth model cannot induce rigid body translation or rotation. We can now express $\boldsymbol{s}(\boldsymbol{x},t)$ as

$$\boldsymbol{s}(\boldsymbol{x},t) = \sum_{\substack{n=1 \\ \omega_n\neq 0}}^{\infty} a_n(t)\, \boldsymbol{s}_n(\boldsymbol{x}), \tag{50}$$

where each of the temporal coefficients $\{a_m(t)\colon \omega_m \neq 0\}$ is obtained by inversion of the associated Laplace transformed coefficient $\{a_m(p)\colon \omega_m \neq 0\}$.

The inversion may be immediately accomplished by an application of the theorem of residues. We shall assume for this purpose that each of the coefficients $f_m(p)$ is a regular function of the complex variable p, except at the origin $p = 0$, where each has, in general, a simple pole. The transient nature of the imposed body force distribution $\boldsymbol{f}(\boldsymbol{x}, t)$ makes this a very natural assumption; in fact, only some very mild restrictions on the behaviour of $\boldsymbol{f}(\boldsymbol{x}, t)$ during the interval $0 \leqslant t \leqslant T$ suffice to guarantee that the expansion coefficients $f_m(p)$ are of this form. Equation (49) shows that each of the coefficients $a_m(p)$ then has three simple poles in the complex p-plane, one at each of $p = 0$ and $p = \pm \mathrm{i}\omega_m$. The residues of $a_m(p)$ at the poles $p = \pm \mathrm{i}\omega_m$ are, respectively, $(\pm 2\mathrm{i}\omega_m)^{-1} f(\pm \mathrm{i}\omega_m)$. The residue at the pole $p = 0$ may be obtained from the Tauberian theorem (Widder 1946) which asserts that

$$\lim_{p\to 0} p\bar{\boldsymbol{f}}(\boldsymbol{x}, p) = \lim_{t\to\infty} \boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{f}_{\mathrm{final}}(\boldsymbol{x}). \tag{51}$$

Making use of the last of equations (46), we find the residue of $a_m(p)$ at the pole $p = 0$ to be simply $\omega_m^{-2} f_m^{\mathrm{final}}$. Closing the Bromwich contour in the right half p-plane for $t < 0$ and in the left half p-plane for $t \geqslant 0$, we obtain

$$a_m(t) = \{\omega_m^{-2} f_m^{\mathrm{final}} + \mathrm{Re}\,[(\mathrm{i}\omega_m)^{-1} f_m(\mathrm{i}\omega_m)\, \mathrm{e}^{\mathrm{i}\omega_m t}]\}\, H(t), \tag{52}$$

where we have made use of the fact that $f_m(-\mathrm{i}\omega_m) = f_m^*(\mathrm{i}\omega_m)$ whenever $\boldsymbol{f}(\boldsymbol{x}, t)$ is real. Insertion of (52) into (50) completely determines the response $\boldsymbol{s}(\boldsymbol{x}, t)$, namely

$$\boldsymbol{s}(\boldsymbol{x}, t) = \sum_{\substack{n=1\\ \omega_n \neq 0}}^{\infty} \{\omega_n^{-2} f_n^{\mathrm{final}} + \mathrm{Re}\,[(\mathrm{i}\omega_n)^{-1} f_n(\mathrm{i}\omega_n)\, \mathrm{e}^{\mathrm{i}\omega_n t}]\}\, \boldsymbol{s}_n(\boldsymbol{x})\, H(t). \tag{53}$$

It is evident from (53) that the response $\boldsymbol{s}(\boldsymbol{x}, t)$ to any real imposed body force distribution will be real, since the basis eigenfunctions $\boldsymbol{s}_n(\boldsymbol{x})$ have been chosen to be either purely real or to occur in complex conjugate pairs.

In any physically realizable Earth model, there will be a dissipative mechanism present which must necessarily lead to a decay of the oscillatory part of the response $\boldsymbol{s}(\boldsymbol{x}, t)$. So long as the amount of dissipation is sufficiently small that it does not appreciably alter the dynamical equations (43), we can determine the ultimate static response $\boldsymbol{s}_{\mathrm{final}}(\boldsymbol{x})$ after the decay of all the normal modes of oscillation independently of any of the details of the dissipative process. We need only assume that some dissipative mechanism exists; this guarantees that the long time limit of $\boldsymbol{s}(\boldsymbol{x}, t)$ exists, and we may thus make use of the Tauberian theorem

$$\lim_{p\to 0} p\bar{\boldsymbol{s}}(\boldsymbol{x}, p) = \lim_{t\to\infty} \boldsymbol{s}(\boldsymbol{x}, t) = \boldsymbol{s}_{\mathrm{final}}(\boldsymbol{x}), \tag{54}$$

applying this result, together with (51), directly to (49). An alternative procedure is to allow for the decay caused by dissipation by simply replacing each purely real eigenfrequency ω_n in equation (53) by the complex expression $\omega_n(1 + \mathrm{i}/2Q_n)$, where $Q_n \gg 1$ is the quality factor of the nth normal mode. In either case, we obtain $a_m^{\mathrm{final}} = \omega_m^{-2} f_m^{\mathrm{final}}$, and therefore

$$\boldsymbol{s}_{\mathrm{final}}(\boldsymbol{x}) = \sum_{\substack{n=1\\ \omega_n \neq 0}}^{\infty} \omega_n^{-2} f_n^{\mathrm{final}}\, \boldsymbol{s}_n(\boldsymbol{x}), \tag{55}$$

a result first pointed out by Gilbert (1971). The result that $\boldsymbol{s}_{\mathrm{final}}(\boldsymbol{x})$ depends only upon $\boldsymbol{f}_{\mathrm{final}}(\boldsymbol{x})$ and not, for example, upon the details of the dissipative process or upon intermediate values of $\boldsymbol{f}(\boldsymbol{x}, t)$ is to be expected. The final configuration of the Earth model after the decay of the free modes of oscillation must be exactly that static configuration which is in mechanical equilibrium with the imposed force distribution $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$. We note that the static response (55) is simply that part of the complete response (53) associated with the pole of the coefficients $a_m(p)$ at the origin $p = 0$; this pole arises directly from the pole of the body force coefficient $f_m(p)$. The dynamic or oscillatory part of the response $\boldsymbol{s}(\boldsymbol{x}, t)$ arises, on the other hand, from the poles of $a_m(p)$ at the points $\pm\mathrm{i}\omega_m$.

An interesting and simple class of imposed body force distributions is the class of all those which have a step function time dependence, i.e. $\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{f}_{\text{final}}(\boldsymbol{x})\, H(t)$, and therefore

$$\bar{\boldsymbol{f}}(\boldsymbol{x}, p) = p^{-1}\boldsymbol{f}_{\text{final}}(\boldsymbol{x}),$$

where $H(t)$ is the unit step function. Equation (53) thus becomes

$$\boldsymbol{s}(\boldsymbol{x}, t) = \sum_{\substack{n=1\\ w_n\neq 0}}^{\infty} \omega_n^{-2} f_n^{\text{final}}\, \boldsymbol{s}_n(\boldsymbol{x})\,[1-\cos\omega_n t]\, H(t), \tag{56}$$

a result, once again, due to Gilbert (1971). Comparison of equations (55) and (56) leads to an interesting physical interpretation of this result. The effect of applying $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ at time $t = 0$ is to shift the equilibrium configuration of the Earth model away from a state of no deformation to the state described by equation (55). Thus, at time $t = 0+$, immediately after the imposition of the body force distribution, the Earth model is no longer in equilibrium, and it starts at that time to oscillate about its new equilibrium configuration.

3. The free oscillations of a rotating Earth model

We will now consider the alterations which must be made in the above theoretical framework in the case of an Earth model which is in mechanical equilibrium not when at rest, but rather when uniformly rotating, with respect to some inertial frame of reference. The discussion will parallel, on a topic-by-topic basis, the discussion of the non-rotating case in §2. It should be borne in mind, and we will point out from time to time, that every property of the non-rotating case is a special version of some more general property of the rotating case. We will first derive the appropriate modifications to the dynamical equations which govern the possible free elastic-gravitational deformations of a uniformly rotating Earth model, and infer a number of properties of the normal mode solutions. We will then consider the excitation of the normal modes of oscillation by a transient imposed body force distribution. In so doing, we must account for the fact that any body force distribution which does not exert a net torque on the Earth model cannot alter its net angular momentum; this implies that we must in general expect that the angular velocity of rotation might be changed as a result of mass redistribution.

The model of the Earth we will consider in this section differs from that considered in § 2 only by the fact that its equilibrium configuration is one of steady diurnal rotation about its centre of mass. We note that any rotating Earth model which is nearly spherically symmetric, and which is rotating sufficiently slowly, can be associated in a well-known way with a unique, corresponding non-rotating Earth model (Jeffreys 1959). The results of this section are by nature abstract, and they do not rely on any such unique correspondence between a rotating Earth model and its non-rotating counterpart; they are therefore not constrained by the preceding assumptions. Any quantitative, numerical comparison of the normal modes of oscillation of non-rotating and rotating Earth models almost certainly would make use of this correspondence (Smith 1974).

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Let $\boldsymbol{\Omega}$ denote the uniform angular velocity of the equilibrium configuration. We will henceforth, *without exception*, adopt the point of view of an observer in the non-inertial frame of reference which, for all times, maintains a state of uniform rotation with constant angular velocity $\boldsymbol{\Omega}$. Let $\hat{\boldsymbol{x}}_1$, $\hat{\boldsymbol{x}}_2$, $\hat{\boldsymbol{x}}_3$ be a Cartesian axis system in this uniformly rotating frame of reference; let the origin $\boldsymbol{O}$ of this Cartesian axis system coincide with the centre of mass of the Earth model, and let $\hat{\boldsymbol{x}}_3$ be aligned along the axis of rotation, so that $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{x}}_3$.

The volume V occupied by the equilibrium configuration and its free exterior surface ∂V are constant domains in the uniformly rotating frame. We continue to denote the location (now measured in the rotating frame) of material particles by $\boldsymbol{x}$, and the unit outward normal to ∂V at the point $\boldsymbol{x}$ by $\hat{\boldsymbol{n}}(\boldsymbol{x})$. As before, $\rho_0(\boldsymbol{x})$ denotes the mass density field, $\phi_0(\boldsymbol{x})$ denotes the gravitational potential field, and $\boldsymbol{T}_0(\boldsymbol{x})$ denotes the equilibrium static stress field. The condition which guarantees mechanical equilibrium of the uniformly rotating configuration is

$$\rho_0(\boldsymbol{x})\,\boldsymbol{\nabla}[\phi_0(\boldsymbol{x})+\psi(\boldsymbol{x})] = \boldsymbol{\nabla}\cdot\boldsymbol{T}_0(\boldsymbol{x}), \tag{57}$$

where $\psi(\boldsymbol{x}) = -\frac{1}{2}[\Omega^2x^2-(\boldsymbol{\Omega}\cdot\boldsymbol{x})^2]$ is the rotational potential due to the apparent centripetal acceleration.

The inertia tensor of this uniformly rotating equilibrium configuration will play a role in the ensuing discussion. We let $\boldsymbol{C}$ denote the inertia tensor as viewed in the uniformly rotating frame of reference, in which case

$$\boldsymbol{C} = \int_V \mathrm{d}V\{\rho_0(\boldsymbol{x})\,[(\boldsymbol{x}\cdot\boldsymbol{x})\,\boldsymbol{I}-\boldsymbol{x}\boldsymbol{x}]\}. \tag{58}$$

We will take it for granted that the axis $\hat{\boldsymbol{x}}_3$ of uniform rotation is the principal axis of greatest inertia of the equilibrium configuration, and we will use $C = \hat{\boldsymbol{x}}_3\cdot\boldsymbol{C}\cdot\hat{\boldsymbol{x}}_3$ to denote the greatest principal moment of inertia. We shall use B and A, respectively, to denote the intermediate and least principal moments of inertia. It is always possible, but not essential to any of the ensuing discussion, to restrict further the alignment of the Cartesian axis system so that $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{x}}_2$, are, respectively, the principal axes of least and intermediate inertia, in which case $A = \hat{\boldsymbol{x}}_1\cdot\boldsymbol{C}\cdot\hat{\boldsymbol{x}}_1$ and $B = \hat{\boldsymbol{x}}_2\cdot\boldsymbol{C}\cdot\hat{\boldsymbol{x}}_2$.

3.1. *The equations of motion governing infinitesimal deformations*

We wish to investigate the possible free, infinitesimal, isentropic, elastic-gravitational deformations which may occur about the uniformly rotating equilibrium configuration. We employ the same mixed Lagrangian–Eulerian description of these deformations as in the non-rotating case, with the exception that the incremental variables $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$, $\rho_1(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{T}}(\boldsymbol{x},t)$ are now measured in the uniformly rotating frame. Once again, we shall apply the principle of least action to an appropriate Lagrangian functional to deduce the dynamical equations of motion and the associated free surface boundary conditions.

The considerations which led to equations (8) and (12) for the elastic and gravitational potential energies accompanying an elastic-gravitational deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ are independent of the presence or absence of any uniform rotation of the equilibrium state. We can thus make immediate use of these expressions for both $\mathfrak{U}[\boldsymbol{s}(\boldsymbol{x},t),\,\phi_1(\boldsymbol{x},t)]$ and $\mathfrak{M}[\boldsymbol{s}(\boldsymbol{x},t),\,\phi_1(\boldsymbol{x},t)]$. The kinetic energy functional $\mathfrak{T}[\boldsymbol{s}(\boldsymbol{x},t),\,\phi_1(\boldsymbol{x},t)]$ is, however, modified by rotation; the uniform rotation of the equilibrium configuration and of the frame of reference necessitates consideration of the rotational kinetic energy associated with that rotation. We will denote the total kinetic energy of the deformed configuration by $\mathfrak{T}_0+\mathfrak{T}[\boldsymbol{s}(\boldsymbol{x},),\,\phi_1(\boldsymbol{x},t)]$, where $\mathfrak{T}_0 = \frac{1}{2}C\Omega^2$ is the kinetic energy of uniform rotation, and $\mathfrak{T}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$ is the incremental kinetic energy associated with the deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$. Dahlen (1973) has shown that

$$\mathfrak{T} = \boldsymbol{\Omega}\cdot\int_V \mathrm{d}V\,[\rho_0\,\boldsymbol{x}\times\partial_t\,\boldsymbol{s}] - \int_V \mathrm{d}V\,[\rho_0 s_j \partial_j \psi] + \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0|\partial_t\,\boldsymbol{s}|^2 - 2\rho_0\,\boldsymbol{s}\cdot(\boldsymbol{\Omega}\times\partial_t\,\boldsymbol{s}) - \rho_0 s_i s_j \partial_i\partial_j\psi]. \tag{59}$$

We will denote by $\mathfrak{T}_{\mathrm{R}}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$ the kinetic energy of deformation measured relative to the uniformly rotating frame;

$$\mathfrak{T}_{\mathrm{R}} = \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0|\partial_t\,\boldsymbol{s}|^2]. \tag{60}$$

In the limit of no rotation, $\boldsymbol{\Omega} = \boldsymbol{0}$ and $\psi(\boldsymbol{x}) = 0$, and the incremental kinetic energy

$$\mathfrak{T}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$$

reduces to the relative kinetic energy $\mathfrak{T}_{\mathrm{R}}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$; this, in turn, is precisely the total kinetic energy (13) in the non-rotating case.

The Lagrangian $\mathfrak{L}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$ is still given by $\mathfrak{L} = \mathfrak{T} - (\mathfrak{U}+\mathfrak{M})$ and, as before, we can simplify it somewhat by appealing to the equilibrium condition (58) and the free surface boundary condition (3). The result is

$$\mathfrak{L} = \boldsymbol{\Omega}\cdot\int_V \mathrm{d}V\,[\rho_0\,\boldsymbol{x}\times\partial_t\,\boldsymbol{s}] + \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0|\partial_t\,\boldsymbol{s}|^2 - 2\rho_0\,\boldsymbol{s}\cdot(\boldsymbol{\Omega}\times\partial_t\,\boldsymbol{s}) - \Lambda_{ijkl}\partial_i s_j \partial_k s_l - 2\rho_0 s_j \partial_j \phi_1 - \rho_0 s_i s_j \partial_i\partial_j(\phi_0+\psi)] - \frac{1}{2}\int_{\mathrm{E}} \mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right]. \tag{61}$$

The form (61) of the Lagrangian $\mathfrak{L}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$ contains a term which is of first order in $\boldsymbol{s}(\boldsymbol{x},t)$, namely

$$\boldsymbol{\Omega}\cdot\int_V \mathrm{d}V\,[\rho_0\,\boldsymbol{x}\times\partial_t\,\boldsymbol{s}] = \boldsymbol{\Omega}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,[\rho_0\,\boldsymbol{x}\times\,\boldsymbol{s}]. \tag{62}$$

This term appears explicitly in the incremental kinetic energy (59) and it represents a valid contribution to that energy; it may, however, be eliminated from the Lagrangian $\mathfrak{L}[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ since, as the left-hand side of (62) implies, its contribution to the action $A[\boldsymbol{s}(\boldsymbol{x},t), \phi_1(\boldsymbol{x},t)]$ exactly vanishes as a consequence of the invariance of the initial and final configurations $\boldsymbol{s}(\boldsymbol{x},t_1)$, $\phi_1(\boldsymbol{x},t_1)$ and $\boldsymbol{s}(\boldsymbol{x},t_2)$, $\phi_1(\boldsymbol{x},t_2)$. Consequently, we may take the Lagrangian appropriate to a uniformly rotating Earth model to have the form

$$\mathfrak{L} = \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0|\partial_t\,\boldsymbol{s}|^2 - 2\rho_0\,\boldsymbol{s}\cdot(\boldsymbol{\Omega}\times\partial_t\,\boldsymbol{s}) - \Lambda_{ijkl}\partial_i s_j \partial_k s_l - 2\rho_0 s_j \partial_j \phi_1 - \rho_0 s_i s_j \partial_i\partial_j(\phi_0+\psi)] - \frac{1}{2}\int_{\mathrm{E}} \mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right]. \tag{63}$$

Note that when $\boldsymbol{\Omega}$ vanishes, the form (63) reduces directly to the Lagrangian (24) for the non-rotating case.

Application of the principle of least action to the Lagrangian (63) leads to the linearized dynamical field equations

$$\left.\begin{aligned} \rho_0\partial_t^2\,\boldsymbol{s} + 2\rho_0\,\boldsymbol{\Omega}\times\partial_t\,\boldsymbol{s} &= -\rho_0\boldsymbol{\nabla}\phi_1 - \rho_0\,\boldsymbol{s}\cdot\boldsymbol{\nabla}[\boldsymbol{\nabla}(\phi_0+\psi)] + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}},\\ \nabla^2\phi_1 &= 4\pi G\rho_1,\\ \rho_1 &= -\boldsymbol{\nabla}\cdot(\rho_0\,\boldsymbol{s}),\\ \tilde{\boldsymbol{T}} &= \boldsymbol{\Lambda}:\boldsymbol{\nabla}\boldsymbol{s}, \end{aligned}\right\} \tag{64}$$

as well as to the associated linearized free surface boundary conditions

$$\left.\begin{aligned} \hat{\boldsymbol{n}}\cdot\tilde{\boldsymbol{T}} &= \boldsymbol{0}, \\ [\phi_1]_-^+ &= 0, \\ [\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_1 + 4\pi G\rho_0\hat{\boldsymbol{n}}\cdot\boldsymbol{s}]_-^+ &= 0. \end{aligned}\right\} \tag{65}$$

The incremental Poisson equation, the linearized continuity equation, and the assumed isentropic constitutive relation are unaffected by rotation, as are the linearized boundary conditions (65). Only the linearized momentum equation, the first of equations (64), is influenced by rotation; there is both a Coriolis force term $2\rho_0(\boldsymbol{x})\,\boldsymbol{\Omega}\times\partial_t\,\boldsymbol{s}(\boldsymbol{x},t)$ on the left-hand side as well as a term depending on the centripetal potential $\psi(\boldsymbol{x})$ on the right-hand side. The most important rotational effects, by far, are due to the presence of the Coriolis force term. The centripetal potential $\psi(\boldsymbol{x})$ appears only in the same combination $\phi_0(\boldsymbol{x})+\psi(\boldsymbol{x})$ which appears in the rotating equilibrium equation (57); it is well known that in the real Earth $\phi_0(\boldsymbol{x})+\psi(\boldsymbol{x})$ differs from $\phi_0(\boldsymbol{x})$ by only about one part in 290 (Jeffreys 1959). The combination $\phi_0(\boldsymbol{x})+\psi(\boldsymbol{x})$ is called the geopotential in the theory of geodesy; its consistent appearance in the rotating dynamical equations in a role equivalent to that played by $\phi_0(\boldsymbol{x})$ alone in the non-rotating case is a reflexion of the fundamental equivalence between inertial mass and gravitational charge.

The appropriate form of the conservation of energy principle for a uniformly rotating Earth model can be obtained, as before, by forming the dot product of the first of equations (64) with the particle velocity vector $\partial_t\,\boldsymbol{s}(\boldsymbol{x},t)$, and integrating the result over V. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int_V \mathrm{d}V\,[\rho_0\,|\partial_t\boldsymbol{s}|^2 + \Lambda_{ijkl}\partial_i s_j\partial_k s_l + 2\rho_0 s_j\partial_j\phi_1 + \rho_0 s_i s_j\partial_i\partial_j(\phi_0+\psi)] + \frac{1}{2}\int_{\mathrm{E}}\mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right]\right\} = 0. \tag{66}$$

This result differs in form from the corresponding result (18) for the non-rotating case only by replacement of the equilibrium gravitational potential $\phi_0(\boldsymbol{x})$ by $\phi_0(\boldsymbol{x})+\psi(\boldsymbol{x})$. A more important difference between the two cases is that the expression in braces in equation (66) is not the sum $\mathfrak{T}+\mathfrak{U}+\mathfrak{M}$ of the kinetic and elastic-gravitational potential energies associated with a freely evolving deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$. In fact we have

$$\begin{aligned} \mathfrak{T}+\mathfrak{U}+\mathfrak{M} = {} & \boldsymbol{\Omega}\cdot\int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x}+\boldsymbol{s})\times\partial_t\,\boldsymbol{s}] - \int_V \mathrm{d}V\,[2\rho_0 s_j\partial_j\psi + \rho_0 s_i s_j\partial_i\partial_j\psi] \\ & + \frac{1}{2}\int_V \mathrm{d}V\,[\rho_0\,|\partial_t\boldsymbol{s}|^2 + \Lambda_{ijkl}\partial_i s_j\partial_k s_l + 2\rho_0 s_j\partial_j\phi_1 + \rho_0 s_i s_j\partial_i\partial_j(\phi_0+\psi)] \\ & + \frac{1}{2}\int_{\mathrm{E}}\mathrm{d}V\left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right]. \end{aligned} \tag{67}$$

It is true, however, that any freely evolving deformation must have $\mathrm{d}(\mathfrak{T}+\mathfrak{U}+\mathfrak{M})/\mathrm{d}t = 0$, and furthermore that equation (66) must be precisely an expression of that fact. To reconcile these results, we must show, from equation (67), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{\Omega}\cdot\int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x}+\boldsymbol{s})\times\partial_t\,\boldsymbol{s}] - \int_V \mathrm{d}V\,[2\rho_0 s_j\partial_j\psi + \rho_0 s_i s_j\partial_i\partial_j\psi]\right\} = 0. \tag{68}$$

It is fairly easy to demonstrate that equation (68) is an expression of the fact that every freely evolving deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ must have $\mathrm{d}\{\boldsymbol{\Omega}\cdot\boldsymbol{H}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]\}/\mathrm{d}t = 0$, where

$\boldsymbol{H}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$ is the associated net angular momentum. It should be emphasized that the terms in braces in equation (68) do represent legitimate contributions to the incremental total energy $\mathfrak{T}+\mathfrak{U}+\mathfrak{M}$ of a free deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$; the law of conservation of angular momentum in the form (68) requires only that the sum of these terms be a constant (but not necessarily vanishing) function of time. This implies that they may be ignored in discussing the time derivative of the incremental total energy. The absence of any Coriolis force term in the expression (66) of the conservation of energy principle is consistent with the well-known fact that the Coriolis force density $2\rho_0(\boldsymbol{x})\ \boldsymbol{\Omega}\times\partial_t\,\boldsymbol{s}(\boldsymbol{x},t)$ is always and everywhere orthogonal to the particle velocity $\partial_t\,\boldsymbol{s}(\boldsymbol{x},t)$, and it can thus do no work.

3.2. *Normal mode solutions and their properties*

We again seek free solutions to the dynamical field equations (64) and (65) in the form of a linear superposition of normal mode solutions, $\boldsymbol{s}(\boldsymbol{x},t)=\boldsymbol{s}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}$, $\phi_1(\boldsymbol{x},t)=\phi_1(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}$. The normal mode eigensolutions $\{\omega,\boldsymbol{s}(\boldsymbol{x}),\phi_1(\boldsymbol{x})\}$ must satisfy the time-independent elastic-gravitational field equations

$$\left.\begin{aligned}-\rho_0\omega^2\boldsymbol{s}+2\mathrm{i}\omega\rho_0\,\boldsymbol{\Omega}\times\boldsymbol{s}&=-\rho_0\boldsymbol{\nabla}\phi_1-\rho_0\,\boldsymbol{s}\cdot\boldsymbol{\nabla}[\boldsymbol{\nabla}(\phi_0+\psi)]+\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}},\\ \nabla^2\phi_1&=4\pi G\rho_1,\\ \rho_1&=-\boldsymbol{\nabla}\cdot(\rho_0\,\boldsymbol{s}),\\ \tilde{\boldsymbol{T}}&=\boldsymbol{\Lambda}:\boldsymbol{\nabla}\boldsymbol{s},\end{aligned}\right\}\tag{69}$$

subject to

$$\left.\begin{aligned}\hat{\boldsymbol{n}}\cdot\tilde{\boldsymbol{T}}&=\boldsymbol{0},\\ [\phi_1]_-^+&=\boldsymbol{0},\\ [\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_1+4\pi G\rho_0\hat{\boldsymbol{n}}\cdot\boldsymbol{s}]_-^+&=0,\end{aligned}\right\}\tag{70}$$

on the free surface ∂V. The Coriolis force term $2\mathrm{i}\omega\rho_0(\boldsymbol{x})\ \boldsymbol{\Omega}\times\boldsymbol{s}(\boldsymbol{x})$ in the first of equations (69) will be seen to induce some profound differences between the properties of normal mode eigensolutions in the rotating and non-rotating cases. One such difference is immediately apparent; the explicit appearance of the factor i makes it clear that we cannot in general restrict attention to purely real eigenfunctions, as we could in the non-rotating case.

We will admit the possibility of complex eigensolutions $\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}$, and we retain the inner product (21) between complex displacement eigenfunctions. As before, we form the inner product of the displacement eigenfunction of a primed eigensolution $\{\omega',\ \boldsymbol{s}'(\boldsymbol{x}),\ \phi_1'(\boldsymbol{x})\}$ with the linearized momentum equation for a different unprimed eigensolution $\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\phi_1(\boldsymbol{x})\}$. We obtain in this case

$$\omega^{*2}\mathscr{T}(\boldsymbol{s}',\boldsymbol{s})-2\omega^*\mathscr{W}(\boldsymbol{s}',\boldsymbol{s})-\mathscr{E}(\boldsymbol{s}',\boldsymbol{s})-\mathscr{G}(\boldsymbol{s}',\boldsymbol{s})-\Phi(\boldsymbol{s}',\boldsymbol{s})=0,\tag{71}$$

where $\mathscr{T}(\boldsymbol{s}',\boldsymbol{s})$, $\mathscr{E}(\boldsymbol{s}',\boldsymbol{s})$ and $\mathscr{G}(\boldsymbol{s}',\boldsymbol{s})$ are defined exactly as before by equations (23) and (24); the form $\mathscr{T}(\boldsymbol{s}',\boldsymbol{s})$ in this case will be called the relative kinetic energy bilinear form, since it is associated with the relative kinetic energy $\mathfrak{T}_{\mathrm{R}}[\boldsymbol{s}(\boldsymbol{x},t),\ \phi_1(\boldsymbol{x},t)]$. The two new bilinear forms in equation (71) are the Coriolis bilinear form $\mathscr{W}(\boldsymbol{s}',\boldsymbol{s})$ and the centripetal bilinear form $\Phi(\boldsymbol{s}',\boldsymbol{s})$, defined by

$$\left.\begin{aligned}\mathscr{W}(\boldsymbol{s}',\boldsymbol{s})&=(\boldsymbol{s}',\mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{s})=\int_V\mathrm{d}V[\rho_0\,\boldsymbol{s}'\cdot(\mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{s})^*],\\ \Phi(\boldsymbol{s}',\boldsymbol{s})&=\int_V\mathrm{d}V[\rho_0 s_i' s_j^*\,\partial_i\partial_j\psi].\end{aligned}\right\}\tag{72}$$

Both are Hermitian symmetric, i.e. $\mathscr{W}(\boldsymbol{s}',\boldsymbol{s})=\mathscr{W}^*(\boldsymbol{s},\boldsymbol{s}')$ and $\Phi(\boldsymbol{s}',\boldsymbol{s})=\Phi^*(\boldsymbol{s},\boldsymbol{s}')$. Reversing the roles of the primed and unprimed eigensolutions gives

$$\omega'^{*2}\mathscr{T}(\boldsymbol{s},\boldsymbol{s}')-2\omega'^*\mathscr{W}(\boldsymbol{s},\boldsymbol{s}')-\mathscr{E}(\boldsymbol{s},\boldsymbol{s}')-\mathscr{G}(\boldsymbol{s},\boldsymbol{s}')-\Phi(\boldsymbol{s},\boldsymbol{s}')=0,\tag{73}$$

and supposing the primed and unprimed eigensolutions to be the same in either of (71) or (73) gives

$$\omega^{*2}\mathscr{T}(\boldsymbol{s},\boldsymbol{s})-2\omega^{*}\mathscr{W}(\boldsymbol{s},\boldsymbol{s})-\mathscr{E}(\boldsymbol{s},\boldsymbol{s})-\mathscr{G}(\boldsymbol{s},\boldsymbol{s})-\Phi(\boldsymbol{s},\boldsymbol{s})=0. \tag{74}$$

Complex conjugation of equation (74) leads to the important result

$$\omega^{2}\mathscr{T}(\boldsymbol{s},\boldsymbol{s})-2\omega\mathscr{W}(\boldsymbol{s},\boldsymbol{s})-\mathscr{E}(\boldsymbol{s},\boldsymbol{s})-\mathscr{G}(\boldsymbol{s},\boldsymbol{s})-\Phi(\boldsymbol{s},\boldsymbol{s})=0, \tag{75}$$

since all of $\mathscr{T}(\boldsymbol{s},\boldsymbol{s})$, $\mathscr{W}(\boldsymbol{s},\boldsymbol{s})$, $\mathscr{E}(\boldsymbol{s},\boldsymbol{s})$, $\mathscr{G}(\boldsymbol{s},\boldsymbol{s})$ and $\Phi(\boldsymbol{s},\boldsymbol{s})$ are real. Equation (75) is the proper generalization of the quadratic equipartition equation (we will show that the name is deserved) to the case of a uniformly rotating Earth model; in the limit of no rotation both $\mathscr{W}(\boldsymbol{s},\boldsymbol{s})$ and $\Phi(\boldsymbol{s},\boldsymbol{s})$ vanish, and equation (75) reduces to the non-rotating equipartition equation (27). Equation (27), in the non-rotating case, allowed the immediate deduction that the squared eigenfrequencies of any non-rotating Earth model, regardless of its constitution, must all be real. Whether the eigenfrequencies were then purely real or purely imaginary depended on considerations of the stability of the equilibrium configuration. Because equation (75), unlike (27), has a term linear in ω which arises from the Coriolis force, we cannot immediately infer that all the squared eigenfrequencies of a rotating Earth model are real, and we must proceed immediately to stability considerations if we wish to make any general remarks about the nature of the possibly complex eigenfrequencies.

It is important, in discussing the dynamics of rotating systems, to distinguish between the concepts of dynamical or ordinary stability, or stability in the absence of dissipative processes, and secular stability, or stability in the presence of dissipative processes (Lamb 1932; Lyttleton 1953). From a practical point of view, the more important notion is that of secular stability, not only because all physically realizable systems are at least slightly dissipative, but also because, as we shall see, secular stability necessarily implies dynamical stability, but not vice versa. The criterion for dynamical stability is that every normal mode eigenfrequency ω must be purely real – exactly the criterion for complete stability in the non-rotating case. Every normal mode eigenfrequency ω of a rotating Earth model must be a root

$$\omega=\mathscr{T}^{-1}[\mathscr{W}\pm(\mathscr{W}^{2}+\mathscr{T}(\mathscr{E}+\mathscr{G}+\Phi))^{\frac{1}{2}}] \tag{76}$$

of the quadratic equation (75). Since both $\mathscr{T}(\boldsymbol{s},\boldsymbol{s})$ and $\mathscr{W}^{2}(\boldsymbol{s},\boldsymbol{s})$ are purely real and positive definite, it is evident from (76) that a simple sufficient condition for dynamical stability is that the sum $\mathscr{E}(\boldsymbol{s},\boldsymbol{s})+\mathscr{G}(\boldsymbol{s},\boldsymbol{s})+\Phi(\boldsymbol{s},\boldsymbol{s})$ be non-negative for all admissible static infinitesimal deformations $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$. This condition is not however necessary; it is only necessary that the discriminant $\mathscr{W}^{2}+\mathscr{T}(\mathscr{E}+\mathscr{G}+\Phi)$ be non-negative in order that every root ω of the form (76) be real.

We shall now show that this sufficient condition for dynamical stability is in fact both a necessary and a sufficient condition for secular stability, provided that some care is taken in defining precisely what is meant by an admissible infinitesimal deformation. Secular stability, or stability in the presence of dissipation, can be investigated by making use of the principle of conservation of energy. Dissipation always tends to decrease the total elastic-gravitational energy content of a freely evolving deformation, and in the presence of dissipation, the conservation of energy principle (66) would be altered to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int_V \mathrm{d}V\,[\rho_0\,|\partial_t\,\boldsymbol{s}|^2+\Lambda_{ijkl}\partial_i s_j\partial_k s_l+2\rho_0 s_j\partial_j\phi_1+\rho_0 s_i s_j\partial_i\partial_j(\phi_0+\psi)]\right.$$
$$\left.+\frac{1}{2}\int_{\mathrm{E}}\mathrm{d}V\left[\frac{1}{4\pi G}|\nabla\phi_1|^2\right]\right\}=\text{a negative quantity}. \tag{77}$$

Consider a rotating Earth model which, beginning at time $t = 0$, is allowed to evolve freely, subject to the energy principle (77), away from the admissible initial configuration $\boldsymbol{s}(\boldsymbol{x}, 0) = \boldsymbol{s}(\boldsymbol{x})$, $\partial_t \boldsymbol{s}(\boldsymbol{x}, 0) = \boldsymbol{0}$, $\phi_1(\boldsymbol{x}, 0) = \phi_1(\boldsymbol{x})$, $\partial_t \phi_1(\boldsymbol{x}, 0) = 0$ (here, *admissible* means only, as before, that $\phi_1(\boldsymbol{x})$ is related to $\boldsymbol{s}(\boldsymbol{x})$ through the incremental Poisson equation and the last two of the boundary conditions (70)). Since the relative kinetic energy term $\mathfrak{T}_{\mathrm{R}}[\boldsymbol{s}(\boldsymbol{x}, t), \phi_1(\boldsymbol{x}, t)]$ within the braces in equation (77) is positive definite, it is clear that the tendency of dissipation will be to restore the deformed Earth model to its equilibrium configuration provided that the sum

$$\frac{1}{2}\int_V \mathrm{d}V\,[\Lambda_{ijkl}\partial_i s_j \partial_k s_l + 2\rho_0 s_j \partial_j \phi_1 + \rho_0 s_i s_j \partial_i \partial_j (\phi_0 + \psi)] + \frac{1}{2}\int_{\mathrm{E}} \mathrm{d}V \left[\frac{1}{4\pi G}|\boldsymbol{\nabla}\phi_1|^2\right] \tag{78}$$

of the remaining terms within the braces in equation (77) is non-negative for the prescribed initial deformation $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$. Consequently, a rotating Earth model will be a secularly stable equilibrium configuration if the sum (78) is a non-negative quantity for every admissible purely real static deformation $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$. This is equivalent to the criterion that $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) + \mathscr{G}(\boldsymbol{s}, \boldsymbol{s}) + \Phi(\boldsymbol{s}, \boldsymbol{s})$ be a non-negative quantity, since $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) + \mathscr{G}(\boldsymbol{s}, \boldsymbol{s}) + \Phi(\boldsymbol{s}, \boldsymbol{s})$ is precisely twice the sum (78) for a purely real static deformation of the form

$$\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})] = \tfrac{1}{2}[\boldsymbol{s}(\boldsymbol{x}) + \boldsymbol{s}^*(\boldsymbol{x})], \quad \mathrm{Re}[\phi_1(\boldsymbol{x})] = \tfrac{1}{2}[\phi_1(\boldsymbol{x}) + \phi_1^*(\boldsymbol{x})];$$

this establishes that $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) + \mathscr{G}(\boldsymbol{s}, \boldsymbol{s}) + \Phi(\boldsymbol{s}, \boldsymbol{s}) \geqslant 0$ is a sufficient condition for secular stability. For it to be a necessary condition as well, we must restrict the class of *admissible* deformations to those which do not alter the angular momentum of the Earth model, since we are here concerned with the secular stability of a freely rotating Earth model which is not subject to any externally imposed forces. It is possible to prescribe initial values

$$\boldsymbol{s}(\boldsymbol{x}, 0) = \mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})], \quad \partial_t \boldsymbol{s}(\boldsymbol{x}, 0) = \boldsymbol{0}, \quad \phi_1(\boldsymbol{x}, 0) = \mathrm{Re}[\phi_1(\boldsymbol{x})], \quad \partial_t \phi_1(\boldsymbol{x}, 0) = 0,$$

which are admissible in the sense that $\mathrm{Re}[\phi_1(\boldsymbol{x})]$ and $\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]$ are related by the incremental Poisson equation and the associated boundary conditions, but which endow the Earth model at the instant $t = 0$ with a net angular momentum different than that of the equilibrium configuration. The restriction on $\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]$ which ensures the invariance of the angular momentum of the Earth model can be shown to be

$$\boldsymbol{\Omega}\cdot\int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x})\,\boldsymbol{x}\times\mathrm{Re}[\boldsymbol{s}(\boldsymbol{x})]] = \boldsymbol{0}. \tag{79}$$

We may thus assert that a necessary and sufficient condition for the secular stability of a freely rotating Earth model is that the inequality $\mathscr{E}(\boldsymbol{s}, \boldsymbol{s}) + \mathscr{G}(\boldsymbol{s}, \boldsymbol{s}) + \Phi(\boldsymbol{s}, \boldsymbol{s}) \geqslant 0$ hold for all admissible complex infinitesimal deformations $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ which satisfy, as well, the constraint (79). As we pointed out earlier, secular stability thus implies dynamical stability.

We note that the criterion for the secular stability of a rotating Earth model is identical to the criterion for the complete stability of a non-rotating Earth model if the static equilibrium gravitational potential $\phi_0(\boldsymbol{x})$ is replaced by the geopotential $\phi_0(\boldsymbol{x}) + \psi(\boldsymbol{x})$, i.e. if we incorporate work done against the centripetal potential $\psi(\boldsymbol{x})$ into the elastic-gravitational potential energy associated with any static deformation. In the limit of no rotation, the criteria for each of complete stability, dynamical stability, and secular stability are the same.

A classical example of a dynamically stable equilibrium configuration which is secularly unstable is that of a rigid body rotating uniformly about its principal axis of least inertia. In the complete absence of dissipation, a truly rigid body will be dynamically stable against small disturbances in this state. Any physically realizable body can only however be quasi-rigid, and there will be dissipative processes acting to decrease the total kinetic energy of rotation. In the absence of any externally imposed forces, the net angular momentum must remain constant, and any quasi-rigid body must ultimately reorient itself so that it is rotating uniformly about its principal axis of greatest inertia; the rate at which this evolution will occur depends on the magnitude of the dissipative mechanism. The state of uniform rotation of a quasi-rigid body about the principal axis of greatest inertia is, on the other hand, both secularly and dynamically stable.

We assume for the remainder of this discussion that the rotating Earth model under consideration is secularly stable. This implies that it will be dynamically stable as well, and hence that all of its normal mode eigenfrequencies ω will be purely real. Such an assumption is, as before, almost certainly correct for any everywhere solid but otherwise 'typical' rotating Earth model. The presence of fluid regions in the Earth model would greatly complicate the question of stability, even more so than in the non-rotating case.

We will now resume the investigation of the properties of the normal mode eigensolutions $\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}$, subject to the constraint that the eigenfrequencies ω must be real. One of the most fundamental properties of the eigensolutions of a non-rotating Earth model is the orthogonality (28) of complex displacement eigenfunctions associated with distinct squared eigenfrequencies. This extremely convenient result is not valid for the more general rotating case. Taking both ω and ω' to be real in (70) and (73) and appealing, as before, to the Hermitian symmetry of the various bilinear forms, we obtain, instead of (28),

$$\mathscr{T}(\boldsymbol{s}', \boldsymbol{s}) - 2(\omega'+\omega)^{-1}\mathscr{W}(\boldsymbol{s}', \boldsymbol{s}) = 0 \quad \text{if} \quad \omega'-\omega \neq 0 \tag{80}$$

or equivalently,

$$(\boldsymbol{s}', \boldsymbol{s}) - 2(\omega'+\omega)^{-1}(\boldsymbol{s}', \mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{s}) = 0 \quad \text{if} \quad \omega'-\omega \neq 0. \tag{81}$$

It is evident that the relation (81) reduces to the conventional orthogonality relation (28) in the limit of no rotation. Conversely, it is clear that orthogonality, as expressed by (28), is in general a property only of non-rotating Earth models. We shall refer to (81) as the quasi-orthogonality relation; it will play a role in the normal mode excitation problem very similar to the role played by the orthogonality condition (28) in the non-rotating case.

In the non-rotating case, we spent some time examining the primitive algebraic structure of the collection of all non-rotating eigensolutions $\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}$. That discussion was founded primarily upon the existence of two distinct reflexion symmetries in the non-rotating normal mode boundary problem (19) and (20). We will now consider a parallel dissection of the collection of all rotating eigensolutions $\{\omega,\ \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$, in order to determine the influence of rotation on the primitive algebraic properties. We wish in particular to show how the algebraic structure of the rotating eigensolutions is related to the somewhat richer structure of the non-rotating eigensolutions, and to exploit this relationship to establish a one-to-one correspondence between the normal modes of oscillation of related rotating and non-rotating Earth models.

It is evident that the presence of a Coriolis force term linear in the angular eigenfrequency ω in the rotating normal mode boundary value problem (69) and (70) destroys one of the two reflexion symmetries enjoyed by the non-rotating case. The eigensolutions of a secularly stable rotating Earth model can only be grouped naturally into pairs; if either of $\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}$ or $\{-\omega,\ \boldsymbol{s}^*(\boldsymbol{x}),\ \phi_1^*(\boldsymbol{x})\}$ is an eigensolution, then both are eigensolutions. We shall disregard, for the moment, the possibility that there might be eigensolutions of a rotating Earth model with associated eigenfunctions which satisfy $\boldsymbol{s}(\boldsymbol{x}) = \alpha\boldsymbol{s}^*(\boldsymbol{x})$, $\phi_1(\boldsymbol{x}) = \alpha\phi_1^*(\boldsymbol{x})$, where α is a complex constant. There are strong physical arguments which suggest that any rotating Earth model can only have two such eigensolutions. Both of these have an associated eigenfrequency $\omega = 0$, and both correspond to a rigid body degree of freedom of the rotating Earth model; these two special secular modes will be treated separately below. The absence of any naturally occurring quartets of eigensolutions in the rotating case implies that every primitive complex eigenspace of displacement eigenfunctions will be of dimension one. These one-dimensional primitive complex eigenspaces can however be grouped naturally in pairs. Associated with every pair of eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ and $\{-\omega, \boldsymbol{s}^*(\boldsymbol{x}), \phi_1^*(\boldsymbol{x})\}$, there is a pair of one-dimensional primitive complex eigenspaces, spanned respectively by $\boldsymbol{s}(\boldsymbol{x})$ and by $\boldsymbol{s}^*(\boldsymbol{x})$.

In the non-rotating case, every primitive complex eigenspace of dimension one or two was associated with, respectively, two eigenfrequencies $\pm\omega$ and a single degree of freedom or four eigenfrequencies, two each of $\pm\omega$, and two degrees of freedom. In the rotating case, the possibilities are more numerous; most naturally occurring pairs of one-dimensional primitive complex eigenspaces will be associated with only two eigenfrequencies $\pm\omega$ and a single degree of freedom, but occasional pairs can be associated with four eigenfrequencies and two degrees of freedom. The equipartition equation (75), in general, provides a means of deciding between these two possibilities. We regard equation (75) as defining a quadratic equation in ω for every complex eigenfunction $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$. The various members of any one-dimensional primitive complex eigenspace all give rise to the same quadratic equation, apart from a real multiplicative constant. Every naturally occurring pair of primitive complex eigenspaces will be associated with two quadratic equipartition equations; each of $\boldsymbol{s}(\boldsymbol{x})$ and $\boldsymbol{s}^*(\boldsymbol{x})$ gives rise to a distinct equation because

$$\mathscr{W}(\boldsymbol{s}^*, \boldsymbol{s}^*) = -\mathscr{W}(\boldsymbol{s}, \boldsymbol{s}),$$

although $$\mathscr{T}(\boldsymbol{s}^*, \boldsymbol{s}^*) = \mathscr{T}(\boldsymbol{s}, \boldsymbol{s}), \quad \mathscr{E}(\boldsymbol{s}^*, \boldsymbol{s}^*) = \mathscr{E}(\boldsymbol{s}, \boldsymbol{s}), \quad \mathscr{G}(\boldsymbol{s}^*, \boldsymbol{s}^*) = \mathscr{G}(\boldsymbol{s}, \boldsymbol{s})$$

and $$\Phi(\boldsymbol{s}^*, \boldsymbol{s}^*) = \Phi(\boldsymbol{s}, \boldsymbol{s}).$$

Every quadratic equipartition equation has two roots ω, and thus every pair of primitive complex eigenspaces can be associated with four distinct roots, namely

$$\omega = \pm\mathscr{T}^{-1}[\mathscr{W} \pm (\mathscr{W}^2 + \mathscr{T}(\mathscr{E} + \mathscr{G} + \Phi))^{\frac{1}{2}}], \tag{82}$$

if we consider multiple roots to be distinct, Every eigenfrequency associated with either member of a pair of one-dimensional primitive complex eigenspaces must be a root (82) of the associated quadratic equipartition equations, but all four roots (82) need not be associated eigenfrequencies. If in fact all four roots (82) associated with some particular pair of one-dimensional primitive complex eigenspaces are associated eigenfrequencies, then we associate all four of those eigenfrequencies and two degrees of freedom with that pair. A much more common situation will be that only two of the roots (82) will be associated eigenfrequencies, namely ω and $-\omega$, and the other two roots will be spurious. We associate only those two eigenfrequencies and a single degree of freedom with any pair of primitive complex eigenspaces which has only two associated valid roots (82). This technique of examining the roots of the associated equipatition equations in order to determine the number of eigenfrequencies and the corresponding number of degrees of freedom to be asigned to any pair of primitive complex eigenspaces is exactly analogous to the technique employed for that purpose in the non-rotating case. The one- and two-dimensional primitive complex eigenspaces of a non-rotating Earth model have, respectively, a single and two identical associated quadratic equipartition equations (27). The roots of any quadratic equation of the form (27) are necessarily of the form $\pm\omega$, and the validity of each is apparent. In the rotating case, we cannot in general determine the validity of all four roots of the form (82) by inspection; each must be directly tested to see if it is an associated eigenfrequency, i.e. a valid solution to the normal mode boundary value problem (69) and (70).

The relation of the normal mode eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ and the corresponding real normal modes of oscillation $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$ of a rotating Earth model to those of any non-rotating counterpart can now be established. It is evident that, in the non-rotating limit, every non-rotating two-dimensional primitive complex eigenspace with four associated eigenfrequencies and two associated degrees of freedom must arise either from the coalescence of two rotating primitive complex eigenspace pairs, each of which has only two associated valid roots and a single associated degree of freedom, or alternatively from the coalescence of a single rotating primitive complex eigenspace pair which has four associated valid roots and two associated degrees of freedom. The effect of rotation on any particular non-rotating two-dimensional primitive complex eigenspace is thus to induce a splitting into two generally distinct pairs of rotating one-dimensional primitive complex eigenspaces. Symbolically,

$$\left.\begin{matrix}\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}\\ \{-\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}\\ \{\omega,\ \boldsymbol{s}^*(\boldsymbol{x}),\ \phi_1^*(\boldsymbol{x})\}\\ \{-\omega,\ \boldsymbol{s}^*(\boldsymbol{x}),\ \phi_1^*(\boldsymbol{x})\}\end{matrix}\right\} \begin{matrix}\nearrow \begin{matrix}\{\omega',\ \boldsymbol{s}'(\boldsymbol{x}),\ \phi_1'(\boldsymbol{x})\}\\ \{-\omega',\ \boldsymbol{s}'^*(\boldsymbol{x}),\ \phi_1'^*(\boldsymbol{x})\}\end{matrix}\\ \searrow \begin{matrix}\{\omega'',\ \boldsymbol{s}''(\boldsymbol{x}),\ \phi_1''(\boldsymbol{x})\}\\ \{-\omega'',\ \boldsymbol{s}''^*(\boldsymbol{x}),\ \phi_1''^*(\boldsymbol{x})\},\end{matrix}\end{matrix} \tag{83}$$

where the arrows denote the splitting effect of rotation. This splitting can occasionally become degenerate if, for example, $\boldsymbol{s}'(\boldsymbol{x})$ and $\boldsymbol{s}''(\boldsymbol{x})$ are not linearly independent; in that case, $\boldsymbol{s}'^*(\boldsymbol{x})$ and $\boldsymbol{s}''^*(\boldsymbol{x})$ would be linearly dependent as well, and the four roots of the two associated quadratic equipartition equations would all be valid eigenfrequencies, namely ω', $-\omega'$, ω'', and $-\omega''$. This degenerate situation will not be common, but we will give two examples of it below.

The eigenfunctions of a non-rotating Earth model which satisfy

$$\boldsymbol{s}(\boldsymbol{x}) = \alpha \boldsymbol{s}^*(\boldsymbol{x}), \quad \phi_1(\boldsymbol{x}) = \alpha \phi_1^*(\boldsymbol{x}),$$

where α is a complex constant, are associated with only a one-dimensional primitive complex eigenspace and a single degree of freedom. The effect of rotation on any such one-dimensional primitive complex eigenspace can only be to induce a splitting into a single pair of rotating primitive complex eigenspaces with two associated valid roots and a single degree of freedom, i.e.

$$\left.\begin{matrix}\{\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}\\ \{-\omega,\ \boldsymbol{s}(\boldsymbol{x}),\ \phi_1(\boldsymbol{x})\}\end{matrix}\right\} \longrightarrow \begin{matrix}\{\omega',\ \boldsymbol{s}'(\boldsymbol{x}),\ \phi_1'(\boldsymbol{x})\}\\ \{-\omega',\ \boldsymbol{s}'^*(\boldsymbol{x}),\ \phi_1'^*(\boldsymbol{x})\}.\end{matrix} \tag{84}$$

Every naturally occurring pair of one-dimensional primitive complex eigenspaces of a rotating Earth model can be used to compose a single real normal mode of oscillation $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$, Any two primitive complex eigenspace pairs which arise by the splitting (83) of a two-dimensional non-rotating primitive complex eigenspace thus can be associated with two real normal modes of oscillation,

$$\left.\begin{aligned}\boldsymbol{s}'(\boldsymbol{x}, t) &= \alpha \boldsymbol{s}'(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega' t} + \alpha^* \boldsymbol{s}'^*(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\omega' t} = 2\,\mathrm{Re}\,[\alpha \boldsymbol{s}'(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega' t}],\\ \phi_1'(\boldsymbol{x}, t) &= \alpha \phi_1'(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega' t} + \alpha^* \phi_1'^*(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\omega' t} = 2\,\mathrm{Re}\,[\alpha \phi_1'(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega' t}]\end{aligned}\right\} \tag{85}$$

and

$$\left.\begin{aligned}\boldsymbol{s}''(\boldsymbol{x}, t) &= \beta \boldsymbol{s}''(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega'' t} + \beta^* \boldsymbol{s}''^*(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\omega'' t} = 2\,\mathrm{Re}\,[\beta \boldsymbol{s}''(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega'' t}],\\ \phi_1''(\boldsymbol{x}, t) &= \beta \phi_1''(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega'' t} + \beta^* \phi_1''^*(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\omega'' t} = 2\,\mathrm{Re}\,[\beta \phi_1''(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\omega'' t}].\end{aligned}\right\} \tag{86}$$

The expressions (85) and (86) are generalized travelling waves which in general have different generalized phase speeds and which do not necessarily travel in exactly opposite directions (except in the accidentally degenerate case where, for example, $\boldsymbol{s}'(\boldsymbol{x})$ and $\boldsymbol{s}''(\boldsymbol{x})$ are not linearly independent and $\omega' = \omega''$). We are thus not, in general, able to construct standing wave representations by a superposition of (85) and (86), except in the non-rotating limit, when both $\boldsymbol{s}'(\boldsymbol{x})$ and $\boldsymbol{s}''(\boldsymbol{x})$ reduce to $\boldsymbol{s}(\boldsymbol{x})$ and both ω' and ω'' reduce to ω. The effect of rotation is to render anisotropic the propagation of the two generalized travelling waves which arise from the splitting (83) of a two-dimensional primitive complex eigenspace. The fundamental eigenfrequency degeneracy of any such two-dimensional primitive complex eigenspace is removed by this anisotropic influence of rotation. We might also expect that rotation will suffice in general to remove as well any higher degree of degeneracy associated with geometrical symmetries of a non-rotating Earth model (although occasional accidental degeneracies can of course always occur). It is well-known that this is so in the case of a non-rotating, sperically symmetric Earth model (Backus & Gilbert 1961).

Any primitive complex eigenspace pair which arises by the splitting (84) of a one-dimensional non-rotating primitive complex eigenspace can be associated with a single real normal mode of oscillation,

$$\left.\begin{aligned} \boldsymbol{s}'(\boldsymbol{x},t) &= \sigma\boldsymbol{s}'(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega' t}+\sigma^*\boldsymbol{s}'^*(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega' t} = 2\,\mathrm{Re}\,[\sigma\boldsymbol{s}'(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega' t}],\\ \phi_1'(\boldsymbol{x},t) &= \sigma\phi_1'(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega' t}+\sigma^*\phi_1'^*(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega' t} = 2\,\mathrm{Re}\,[\sigma\phi_1'(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega' t}].\end{aligned}\right\} \tag{87}$$

The expression (87) is a generalized travelling wave which, in the limit of no rotation, when $\boldsymbol{s}'(\boldsymbol{x})$ reduces to $\boldsymbol{s}(\boldsymbol{x})$ and ω' reduces to ω, becomes a generalized travelling wave with vanishing phase speed, i.e. a standing wave, since $\boldsymbol{s}(\boldsymbol{x})$ and $\boldsymbol{s}^*(\boldsymbol{x})$ are not linearly independent. In general, the normal modes of free oscillation of a rotating Earth model will be generalized travelling waves of the form (85), (86) or (87); such representations do not have the property that the particle motion $\boldsymbol{s}(\boldsymbol{x},t)$ and the associated incremental gravitational potential $\phi_1(\boldsymbol{x},t)$ are everywhere in V either in phase, or exactly out of phase by π radians. Every independent normal mode of oscillation of the form (85), (86) or (87) is associated with a single degree of freedom of a rotating Earth model; a Cauchy initial value prescription is both necessary and sufficient for the determination of the complex constants α, β or σ which are required for the complete specification of any individual mode.

Equation (75) is the generalization of the non-rotating equipartition equation (27) to the rotating case. It is easy to show that equation (75) is an expression of the fact that, over any integral number of cycles, the averaged incremental kinetic energy $\mathfrak{T}[\boldsymbol{s}(\boldsymbol{x},t),\,\phi_1(\boldsymbol{x},t)]$ associated with any of the normal modes of oscillation (85), (86) or (87) is equal to the averaged elastic-gravitational potential energy $\mathfrak{U}[\boldsymbol{s}(\boldsymbol{x},t),\,\phi_1(\boldsymbol{x},t)]+\mathfrak{M}[\boldsymbol{s}(\boldsymbol{x},t),\,\phi_1(\boldsymbol{x},t)]$. This justifies the use of the term *equipartition equation* in referring to equation (75). The additional terms $2\omega\mathscr{W}(\boldsymbol{s},\boldsymbol{s})$ and $\Phi(\boldsymbol{s},\boldsymbol{s})$ in (75) which do not appear in (27) are contributions to the averaged incremental kinetic energy which arise directly from the uniform rotation of the Earth model. The second of these terms, $\Phi(\boldsymbol{s},\boldsymbol{s})$, can also be interpreted as arising from the averaged work done against the centripetal potential field $\psi(\boldsymbol{x})$ during a normal mode of oscillation; its appearance in the equipartition equation (75) is consistent with the replacement of the equilibrium gravitational potential $\phi_0(\boldsymbol{x})$ by the geopotential $\phi_0(\boldsymbol{x})+\psi(\boldsymbol{x})$. The first of these terms, $2\omega\mathscr{W}(\boldsymbol{s},\boldsymbol{s})$, is associated with the Coriolis force; it is, in fact, the time average of a quantity which does not change with time. The rotational kinetic energy associated with the Coriolis force is a genuine contribution to the total energy budget of any elastic-gravitational normal mode of oscillation (85), (86) or (87) of a rotating Earth model, but it can be readily shown to have a vanishing time derivative. Thus in the absence of dissipation, energy initially stored as Coriolis energy always remains Coriolis energy. This is consistent with the well known and previously mentioned observation that the Coriolis force can do no work; it is for precisely this reason that a Coriolis force term does not appear in the conservation of energy principle (66).

As in the non-rotating case, the equipartition equation is associated with a useful variational principle known as Rayleigh's principle. We regard the left-hand side of (75) as defining a functional $\omega^2\mathscr{T}(\boldsymbol{s},\boldsymbol{s})-2\omega\mathscr{W}(\boldsymbol{s},\boldsymbol{s})-\mathscr{E}(\boldsymbol{s},\boldsymbol{s})-\mathscr{G}(\boldsymbol{s},\boldsymbol{s})-\Phi(\boldsymbol{s},\boldsymbol{s})$ of the elastic-gravitational disturbance $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$. Rayleigh's principle asserts, as before, that this functional is stationary under independent small variations $\delta\boldsymbol{s}(\boldsymbol{x})$, $\delta\phi_1(\boldsymbol{x})$ if and only if $\boldsymbol{s}(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$ is an elastic-gravitational eigenfunction with eigenfrequency ω. This principle is an immediate consequence of the Hermitian symmetry of the bilinear forms $\mathscr{T}(\boldsymbol{s}',\boldsymbol{s})$, $\mathscr{W}(\boldsymbol{s}',\boldsymbol{s})$, $\mathscr{E}(\boldsymbol{s}',\boldsymbol{s})$, $\mathscr{G}(\boldsymbol{s}',\boldsymbol{s})$ and $\Phi(\boldsymbol{s}',\boldsymbol{s})$, or it may alternatively be shown to follow directly from the principle of least action.

Every non-rotating Earth model has at least six zero-frequency or secular normal modes of oscillation, corresponding to the rigid body translations and rotations of that Earth model in an inertial frame of reference. We will now take up an examination of the effects of rotation on these rigid body secular modes. The 6 degrees of freedom of rigid body motion give rise, in a non-rotating Earth model, to a 12-dimensional degenerate complex eigenspace of displacement eigenfunctions with 12 associated eigenfrequencies, all of which vanish. A rotating Earth model must also have 6 degrees of freedom and 12 associated eigenfrequencies which correspond, in some sense, to rigid body motions. We pursue this question by making use of the representations (38), (39), (40) and (41) of the non-rotating one- and two-dimensional primitive complex eigenspaces of rigid body translational and rotational displacement eigenfunctions.

We consider first the effect of rotation on the one-dimensional primitive complex eigenspace (38) corresponding to rigid body translations parallel to the rotation axis $\hat{\boldsymbol{x}}_3$. It is readily verified that (38) remains an eigenfunction of any rotating Earth model, with an associated eigenfrequency $\omega=0$. The technique of examining the roots of the associated equipartition equation (75) in order to determine the number of eigenfrequencies, and thus the number of degrees of freedom, to be associated with any particular eigenfunction is not strictly valid in the case of an eigenfunction with an associated eigenfrequency $\omega=0$, but it can easily be otherwise verified in this case that the eigenfunction (38) does correspond to a single rigid body degree of freedom, and has therefore two associated, identical, vanishing eigenfrequencies. This particular normal mode of oscillation, which we shall call the *axial translational mode,* is thus completely unaffected by rotation. Note that the displacement eigenfunction (38) has the property that $\boldsymbol{s}(\boldsymbol{x})$ and its complex conjugate $\boldsymbol{s}^*(\boldsymbol{x})$ are not linearly independent; the zero-frequency eigenspace associated with the axial translational mode is in fact one of the two eigenspaces mentioned earlier as having this property. The axial translational mode is purely secular, and the associated real deformation $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ is of the form

$$\left.\begin{aligned}\boldsymbol{s}(\boldsymbol{x},t)&=[\hat{\boldsymbol{r}}Y_1^0(\theta,\phi)+\boldsymbol{\nabla}_1Y_1^0(\theta,\phi)]\,[E+Ft],\\ \phi_1(\boldsymbol{x},t)&=-\boldsymbol{s}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\phi_0(\boldsymbol{x}),\end{aligned}\right\}\tag{88}$$

where E and F are arbitrary real constants.

The effect of rotation on the one-dimensional primitive complex eigenspace (40) corresponding to rigid body rotations about the rotation axis $\hat{\boldsymbol{x}}_3$ is somewhat more complex. We can verify, as in the previous case, that (40) remains an eigenfunction of a rotating Earth model, with associated eigenfrequency $\omega=0$. In the present instance however, and in contrast to the previous case, it is apparent that the eigenfunction (40) cannot by itself, correspond to the rigid body degree of freedom associated with rotation about the $\hat{\boldsymbol{x}}_3$ axis. The real deformation $\boldsymbol{s}(\boldsymbol{x}, t)$, $\phi_1(\boldsymbol{x}, t)$ associated with any normal mode of a rotating Earth model requires a Cauchy initial value prescription for its complete determination. The secular eigenfunction (40) can clearly only be associated with a part of any Cauchy initial value prescription, namely the prescribed initial displacement $\boldsymbol{s}(\boldsymbol{x}, 0)$, $\phi_1(\boldsymbol{x}, 0)$. Any non-zero prescribed initial velocity $\partial_t \boldsymbol{s}(\boldsymbol{x}, 0)$, $\partial_t \phi_1(\boldsymbol{x}, 0)$ will in general give rise to an associated infinitesimal change in the rate of angular rotation about the $\hat{\boldsymbol{x}}_3$ axis; this change in the rate of angular rotation of any deformable (i.e. not rigid) Earth model cannot be accommodated without some additional deformation (in general, an equatorial bulge) in response to the associated incremental rotational potential. We will call the secular mode associated with the rigid body degree of freedom of rotation about the $\hat{\boldsymbol{x}}_3$ axis the axial spin mode. Its associated real deformation will, in general, be of the form

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x}, t) &= [-r\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^0(\theta, \phi)]\,[E+Ft] + [\boldsymbol{s}_{\text{a.s.}}(\boldsymbol{x})]\,F, \\ \phi_1(\boldsymbol{x}, t) &= [r\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_1 Y_1^0(\theta, \phi) \cdot \boldsymbol{\nabla}\phi_0(\boldsymbol{x})]\,[E+Ft] + [\phi_{1\text{a.s.}}(\boldsymbol{x})]\,F, \end{aligned}\right\} \tag{89}$$

where, once again, E and F are arbitrary real constants. A Cauchy initial value prescription is both necessary and sufficient for the determination of these constants E and F. The constant E multiplies the eigenfunction (40); this term represents in (89) nothing more than a constant angular offset of the Earth away from its equilibrium position. The functions $\boldsymbol{s}_{\text{a.s.}}(\boldsymbol{x})$ and $\phi_{1\text{a.s}}(\boldsymbol{x})$ have been used to denote that part of the deformation which is associated with the response to the incremental rotational potential; the precise form of this response depends, of course, on the various material properties $\rho_0(\boldsymbol{x})$ and $\Lambda(\boldsymbol{x})$ of the Earth model. We note that $\boldsymbol{s}_{\text{a.s.}}(\boldsymbol{x})$, $\phi_{1\text{a.s.}}(\boldsymbol{x})$ is *not* a normal mode eigenfunction with an associated eigenfrequency $\omega = 0$, although (89) is of course a solution to the time-dependent system of equations (64) and (65). The other term appearing in (89) is (40) and it of course is an eigenfunction; it is in fact the second of the above mentioned eigenfunctions which has the property that $\boldsymbol{s}(\boldsymbol{x})$ and $\boldsymbol{s}^*(\boldsymbol{x})$ are not linearly independent. Intuition suggests strongly that (38) and (40) will be the only eigenfunctions of any rotating Earth model which have this property. The inherently anisotropic influence of rotation should guarantee that any other normal mode of oscillation which is in any way affected by rotation (and, in general, all the other normal modes will be so affected) will be associated with a pair of one-dimensional primitive complex eigenspaces which are truly distinct.

We consider next the effect of rotation on the two-dimensional primitive complex eigenspace (39), corresponding to rigid body translation orthogonal to the rotation axis $\hat{\boldsymbol{x}}_3$. It is readily verified that any poloidal vector field of the form (39), with $\beta = 0$, is a displacement eigenfunction of a rotating Earth model, with associated eigenfrequency $\omega = \Omega$, and that any poloidal vector field of the form (39), with $\alpha = 0$, is likewise a displacement eigenfunction, with associated eigenfrequency $\omega = -\Omega$. Examination of the associated versions of the equipartition equation (75) reveals that each of Ω and $-\Omega$ is in fact a double eigenfrequency. We have in this case an example of the degenerate splitting of a two-dimensional primitive complex eigenspace into a single pair of one-dimensional complex eigenspaces, each of which is associated with two eigenfrequencies; in this particular example, the effect of rotation on the eigenfunctions is nil and the two associated eigenfrequencies are identical (i.e. in the notation of (83), $\{\omega', \boldsymbol{s}'(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ and $\{\omega'', \boldsymbol{s}''(\boldsymbol{x}), \phi_1''(\boldsymbol{x})\}$ are identical, and furthermore $\boldsymbol{s}'(\boldsymbol{x})$ and $\boldsymbol{s}''(\boldsymbol{x})$ are both equal to $\boldsymbol{s}(\boldsymbol{x})$). Each of the one-dimensional primitive complex eigenspaces (i.e. (39) with $\beta = 0$ and $\alpha = 0$, respectively) gives rise to an associated time-dependent displacement field $\boldsymbol{s}(\boldsymbol{x}, t)$ whose orientation is fixed in an inertial frame of reference, and thus in the uniformly rotating frame of reference, appears to rotate backward, i.e. in a direction opposite to that of the rotation of the Earth model, at exactly the rotation rate Ω. We will call the normal modes of oscillation which describe this class of motions the *equatorial translational modes*; the most general form for an equatorial translational mode of oscillation is

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= (\alpha+\beta t)\,\mathrm{e}^{\mathrm{i}\Omega t}\,[\hat{\boldsymbol{r}}Y_1^1(\theta,\phi)+\boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)] \\ &\quad +(\alpha^*+\beta^* t)\,\mathrm{e}^{-\mathrm{i}\Omega t}[\hat{\boldsymbol{r}}Y_1^1(\theta,\phi)+\boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)]^* \\ &= 2\,\mathrm{Re}\,\{(\alpha+\beta t)\,\mathrm{e}^{\mathrm{i}\Omega t}\,[\hat{\boldsymbol{r}}Y_1^1(\theta,\phi)+\boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)]\}, \\ \phi_1(\boldsymbol{x},t) &= -\boldsymbol{s}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\phi_0(\boldsymbol{x}), \end{aligned}\right\} \tag{90}$$

where α and β are complex constants. The particle motion $\boldsymbol{s}(\boldsymbol{x},t)$ described by (90) has the somewhat unusual form of an offset linear spiral parameterized by time. The secular factor $\alpha+\beta t$ in (90) arises because, as seen in an inertial frame of reference, the equatorial translational modes are still truly secular; the non-zero double eigenfrequencies $\omega = \pm\Omega$ arise solely from the uniform rotation of the adopted coordinate system. Every rotating Earth model has two degrees of freedom associated with the equatorial translational modes, and a Cauchy initial value prescription is both necessary and sufficient for the determination of the two complex constants in (90).

We consider finally the effect of rotation on the remaining rigid body eigenspace (41), corresponding to rigid body rotations about axes orthogonal to the rotation axis $\hat{\boldsymbol{x}}_3$. It is readily verified that any toroidal vector field of the form (41), with $\beta = 0$, is a displacement eigenfunction of a rotating Earth model, with associated eigenfrequency $\omega = \Omega$, and that any toroidal vector field of the form (41), with $\alpha = 0$, is likewise a displacement eigenfunction, with associated eigenfrequency $\omega = -\Omega$. The one-dimensional primitive complex eigenspace corresponding to $\beta = 0$ can be shown to yield a version of the equipartition equation whose roots are the associated eigenfrequency $\omega = \Omega$ and the generally spurious (but see below) root

$$\omega = -\Omega[C-\tfrac{1}{2}(A+B)]/[\tfrac{1}{2}(A+B)],$$

where A, B and C are the least, intermediate, and greatest principal moments of inertia of the Earth model. The one-dimensional primitive complex eigenspace corresponding to $\alpha = 0$ leads to similar roots having opposite signs, i.e. $\omega = -\Omega$ and $\omega = \Omega[C-\frac{1}{2}(A+B)]/[\frac{1}{2}(A+B)]$. The existence of the two spurious roots associated with the pair of one-dimensional primitive complex eigenspaces, (41) with $\beta = 0$ and $\alpha = 0$, is an indication that the non-rotating two-dimensional primitive complex eigenspace (41) in general suffers a non-degenerate splitting (i.e. in the notation of (83), $\{\omega', \boldsymbol{s}'(\boldsymbol{x}), \phi_1'(\boldsymbol{x})\}$ and $\{\omega'', \boldsymbol{s}''(\boldsymbol{x}), \phi_1''(\boldsymbol{x})\}$ are distinct). The pair of one-dimensional primitive complex eigenspaces defined by (41) with $\beta = 0$ and $\alpha = 0$ have a single associated degree of freedom, and an associated time-dependent normal mode of oscillation of the form

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \alpha(-r\hat{\boldsymbol{r}}\times\boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\Omega t})+\alpha^*(-r\hat{\boldsymbol{r}}\times\boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}\Omega t})^* \\ &= 2\,\mathrm{Re}\,[-\alpha r\hat{\boldsymbol{r}}\times\boldsymbol{\nabla}_1 Y_1^1(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\Omega t}], \\ \phi_1(\boldsymbol{x},t) &= -\boldsymbol{s}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\phi_0(\boldsymbol{x}), \end{aligned}\right\} \tag{91}$$

where α is an arbitrary complex constant. The motion (91) amounts simply to a tilt of the actual axis of rotation of the Earth model away from $\hat{\boldsymbol{x}}_3$, the axis of rotation of the uniformly rotating frame of reference; we will call this mode the *tilt-over mode*. We note that in a reference frame which is attached to the Earth model in such a way that the internal angular momentum is prescribed to vanish (i.e. in Tisserand's mean axes of body; Munk & MacDonald 1960), this mode would necessarily appear to have zero amplitude at all times.

Since the non-rotating two-dimensional primitive complex eigenspace (41) has two associated degrees of freedom, and the pair of one-dimensional primitive complex eigenspaces associated with the tilt-over mode has but one, it is apparent that the above picture of the rigid body motions of a rotating Earth model is incomplete. In fact, the remaining rigid body degree of freedom is in general associated with the Eulerian free nutation of the Earth model; the Eulerian free nutation of the real Earth is commonly called the Chandler wobble after its discoverer (Munk & MacDonald 1960). Every rotating Earth model which is not everywhere fluid will have a pair of one-dimensional primitive complex eigenspaces, spanned by, say, $\boldsymbol{s}_{\mathrm{C.w.}}(\boldsymbol{x})$ and $\boldsymbol{s}^*_{\mathrm{C.w.}}(\boldsymbol{x})$, respectively, with associated gravitational potential perturbations $\phi_{1\mathrm{C.w.}}(\boldsymbol{x})$ and $\phi^*_{1\mathrm{C.w.}}(\boldsymbol{x})$, and associated eigenfrequencies $\omega_{\mathrm{C.w.}}$ and $-\omega_{\mathrm{C.w.}}$. This pair of Chandler wobble eigenspaces will have an associated Chandler wobble normal mode of oscillation,

$$\left.\begin{aligned} \boldsymbol{s}(\boldsymbol{x},t) &= \alpha\boldsymbol{s}_{\mathrm{C.w.}}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{C.w.}}t}+\alpha^*\boldsymbol{s}^*_{\mathrm{C.w.}}(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{C.w.}}t}\\ &= 2\,\mathrm{Re}\,[\alpha\boldsymbol{s}_{\mathrm{C.w.}}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{C.w.}}t}],\\ \phi_1(\boldsymbol{x},t) &= \alpha\phi_{1\mathrm{C.w.}}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{C.w.}}t}+\alpha^*\phi^*_{1\mathrm{C.w.}}(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{C.w.}}t}\\ &= 2\,\mathrm{Re}\,[\alpha\phi_{1\mathrm{C.w.}}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{C.w.}}t}]. \end{aligned}\right\} \tag{92}$$

For any 'typical' Earth model, the associated Chandler wobble period $2\pi/|\omega_{\mathrm{C.w.}}|$ will be on the order of fourteen months. The Chandler wobble eigenfunctions $\boldsymbol{s}_{\mathrm{C.w.}}(\boldsymbol{x})$, $\phi_{1\mathrm{C.w.}}(\boldsymbol{x})$ and $\boldsymbol{s}^*_{\mathrm{C.w.}}(\boldsymbol{x})$, $\phi^*_{1\mathrm{C.w.}}(\boldsymbol{x})$ of any 'typical' Earth model cannot be readily expressed without rather exetnsive numerical computation because of the complications arising from elasticity and gravitation. The classical approximation, due to Love (1909) and Larmor (1909), is based on the assumptions that the Earth model responds quasi-statically and as if it were spherically symmetric to the incremental centripetal potential associated with any shift of its rotation axis. Smith (1974) has proposed a computational scheme which avoids these rather restrictive assumptions, and which should permit an accurate numerical computation of the elastic-gravitational Chandler wobble eigenfunctions $\boldsymbol{s}_{\mathrm{C.w.}}(\boldsymbol{x})$, $\phi_{1\mathrm{C.w.}}(\boldsymbol{x})$ and $\boldsymbol{s}^*_{\mathrm{C.w.}}(\boldsymbol{x})$, $\phi^*_{1\mathrm{C.w.}}(\boldsymbol{x})$ and the associated eigenfrequencies $\omega_{\mathrm{C.w.}}$ and $-\omega_{\mathrm{C.w.}}$ for any 'typical' slowly rotating Earth model which is in hydrostatic equilibrium. The Chandler wobble normal mode of oscillation of a rotating Earth model differs from the preceding five rigid body modes in that its associated temporal behaviour $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ is completely non-secular. A further distinction is that in case of any physically realizable Earth model with imperfections of elasticity, the amplitude of the Chandler wobble will decay with time, but the amplitude of the other rigid body modes will not. The splitting of the two equatorial rigid body rotational modes of a non-rotating Earth model into the purely rigid body tilt-over mode and the Chandler wobble or Eulerian free nutational mode of its rotating counterpart provides an interesting example of the inherently anisotropic influence of rotation. The tilt-over mode of oscillation (91) is a generalized travelling wave which propagates backward, i.e. in a direction opposite to that of the rotation of the Earth model, with a generalized phase speed of one equatorial revolution per day, while the Chandler wobble mode of oscillation (92) is a generalized travelling wave which, at least for any 'typical' Earth model, propagates forward with a generalized phase speed of one equatorial revolution every 14 months.

The class of rotating Earth models which are perfectly rigid, and thus incapable of any internal deformation, provides a particularly simple case for which the Chandler wobble eigensolutions are well known. The Chandler wobble or Eulerian free nutation of any rigid Earth model has a pair of associated one-dimensional primitive complex eigenspaces consisting, respectively, of toroidal vector fields of the form (41) with

$$\frac{\alpha}{\beta}=\left(1\mp\sqrt{\frac{A(C-B)}{B(C-A)}}\right)\bigg/\left(1\pm\sqrt{\frac{A(C-B)}{B(C-A)}}\right), \tag{93}$$

where A, B and C are, as before, the least, intermediate, and greatest principal moments of inertia. The associated eigenfrequencies are, respectively,

$$\omega_{\mathrm{C.w.}} = \pm\sqrt{\left(\frac{C-A}{A}\frac{C-B}{B}\right)}\,\Omega. \tag{94}$$

An interesting accidental degeneracy occurs in the case of a rotating Earth model which is not only rigid, but also axially symmetric in the sense that $A = B$. In that case, the rigid body Eulerian nutational eigenfrequencies (94) coincide with the generally spurious roots of the quadratic equipartition equations defined by the tilt-over mode eigenfunctions, (41) with $\beta = 0$ and $\alpha = 0$. The splitting of the two-dimensional primitive complex eigenspace corresponding to the equatorial rigid body rotations is thus in this one accidental instance of the degenerate variety; inspection of equations (93) shows clearly that the Chandler wobble eigenfunctions of any rigid, axially symmetric rotating Earth model coincide exactly with the tilt-over mode eigenfunctions. The Eulerian free nutation of an arbitrarily triaxial rigid Earth model is in general elliptically polarized, but for an axially symmetric Earth model, it is circularly polarized.

3.3. *Normal mode excitation by a transient source*

We consider now the excitation of the elastic-gravitational free oscillations of a freely rotating Earth model by the action of an imposed body force distribution. We suppose that, prior to time $t = 0$, the Earth model is in equilibrium and rotating uniformly about its centre of mass **O**, with constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{x}}_3$ with respect to some inertial frame of reference. The net angular momentum $\boldsymbol{H}$ of this uniformly rotating equilibrium configuration is $\boldsymbol{H} = C\Omega\hat{\boldsymbol{x}}_3$, where $C = \hat{\boldsymbol{x}}_3\cdot\boldsymbol{C}\cdot\hat{\boldsymbol{x}}_3$ is the greatest principal moment of inertia. As before, we shall consider only those body force distributions which could be used to represent some physically realizable process which is wholly internal to the Earth, e.g. an earthquake source mechanism. We will therefore prescribe the obvious and natural constraints that the imposed body force distribution can exert neither a net force nor a net torque on the Earth model. These constraints will assure that the centre of mass **O** of the deformed Earth model remains fixed in the appropriate inertial frame of reference, and that the angular momentum vector $\boldsymbol{H}$ of the Earth model will remain constant. We will maintain throughout the ensuing discussion the viewpoint of an observer situated in the uniformly rotating frame of reference with origin at $\boldsymbol{O}$ and constant angular velocity of rotation $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{x}}_3$.

In the present instance, some care is required in choosing an appropriate analytical description of an imposed body force distribution which is meant to represent a process 'wholly internal' to the Earth. The reason for this is connected with the existence of the secular normal modes of a rotating Earth model. As previously noted, we have restricted consideration in this paper to those Earth models whose secular mode structure is associated only with rigid body degrees of freedom of the entire Earth model. A rotating Earth model of this type has only four secular normal modes: a single axial and two equatorial translational modes (88) and (90), and an axial spin mode (89). The remaining two modes associated with the six rigid body degrees of freedom, the tilt-over mode (91) and the Chandler wobble (92), are not secular in character. Excitation of any translational mode necessitates imparting linear momentum to the Earth model, and the constraint that the imposed body force distribution can exert no net force on the Earth model clearly prohibits excitation of these modes. Similarly, excitation of the non-secular tilt-over mode requires an alteration in the direction of the angular momentum vector of the Earth model, and body force distributions which exert no net torque on the Earth model will not excite this mode. The other two normal modes of a rotating Earth model associated with the rigid body degrees of freedom can, on the other hand, be excited, even by a 'wholly internal' imposed body force distribution with zero net force and torque. The Chandler wobble is not a secular normal mode, and it can be treated in an excitation problem exactly like any other non-secular elastic-gravitational normal mode. The axial spin mode is a secular normal mode, and consequently we shall have to make special provision to account for it.

In general, the deformation arising from an internally imposed body force distribution may alter the inertia tensor (and, in particular, the principal moment of greatest inertia) of the Earth model. Conservation of angular momentum will then require a compensating change in the Earth model's angular speed of rotation. The difference between the old and the new rotation rate will be infinitesimally small but will, for sufficiently long times, lead to an arbitrarily large and secularly growing deviation between the deformed Earth model and its equilibrium configuration. As already pointed out in the non-rotating case, large deviations away from equilibrium are not formally compatible with a conventional linearized theory of infinitesimal oscillations. This difficulty is exactly the result of excitation of the secular axial spin mode by the imposed body force distribution.

We can accommodate this possibility with minimal disruption of our extant theoretical framework by explicitly and separately allowing for a change in the rate of angular rotation of the Earth model. We note that the most general forms of the Lagrangian particle position $\boldsymbol{r}(\boldsymbol{x},t)$ and the Eulerian total gravitational potential $\phi_{\mathrm{E}}(\boldsymbol{r},t)$ associated with the axial spin mode are

$$\left.\begin{aligned} \boldsymbol{r}(\boldsymbol{x},t) &= \boldsymbol{Q}(t)\cdot[\boldsymbol{x}+\boldsymbol{s}_{\mathrm{a.s.}}(\boldsymbol{x})], \\ \phi_{\mathrm{E}}(\boldsymbol{r}(\boldsymbol{x},t),t) &= \phi_0(\boldsymbol{x}+\boldsymbol{s}_{\mathrm{a.s.}}(\boldsymbol{x}))+\phi_{1\mathrm{a.s.}}(\boldsymbol{Q}(t)\cdot\boldsymbol{x}), \end{aligned}\right\} \tag{95}$$

where $\boldsymbol{Q}(t)$ is a proper orthogonal tensor, with Cartesian components relative to $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ of the form

$$\boldsymbol{Q}(t) = \begin{pmatrix} \cos(E+Ft) & \sin(E+Ft) & 0 \\ -\sin(E+Ft) & \cos(E+Ft) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{96}$$

where E and F are, as in equations (89), arbitrary real constants. The expressions (89) for the incremental perturbations $\boldsymbol{s}(\boldsymbol{x},t)$, $\phi_1(\boldsymbol{x},t)$ associated with the axial spin mode are simply a linearized version of the uniformly valid expressions (95) and (96). In a general excitation problem, we can allow for the excitation of the various non-secular modes, as well as the secular axial spin mode, by representing the Lagrangian particle position $\boldsymbol{r}(\boldsymbol{x},t)$ and the associated Eulerian total gravitational potential $\phi_{\mathrm{E}}(\boldsymbol{r},t)$ in the forms

$$\left.\begin{aligned} \boldsymbol{r}(\boldsymbol{x},t) &= \boldsymbol{Q}(t)\cdot[\boldsymbol{x}+\boldsymbol{s}(\boldsymbol{x},t)], \\ \phi_{\mathrm{E}}(\boldsymbol{r}(\boldsymbol{x},t),t) &= \phi_0(\boldsymbol{x}+\boldsymbol{s}(\boldsymbol{x},t))+\phi_1(\boldsymbol{Q}(t)\cdot\boldsymbol{x},t), \end{aligned}\right\} \tag{97}$$

where $\boldsymbol{Q}(t)$ is of the form (96). The forms (97) are direct generalizations of the forms (4) and (7), but allowing for a finite rigid rotation of the deformed Earth model away from its equilibrium configuration. The initially uniform rotation, for times $t<0$, implies that $\boldsymbol{Q}(t)=\boldsymbol{I}$ for $t<0$ and, thus that the constant E vanishes identically in the present case. It is convenient to relabel the constant F (which is simply the incremental change in the rate of angular rotation) by $\delta\Omega$ and to introduce the associated incremental change $\Delta\boldsymbol{\Omega}(t)$ in the instantaneous angular velocity of rotation,

$$\Delta\boldsymbol{\Omega}(t) = \delta\Omega\hat{\boldsymbol{x}}_3 H(t). \tag{98}$$

The initial conditions on $\boldsymbol{s}(\boldsymbol{x},t)$ and $\phi_1(\boldsymbol{x},t)$ consistent with the initially uniform rigid rotation are, as before, that $\boldsymbol{s}(\boldsymbol{x},0)=\boldsymbol{0}$, $\partial_t\boldsymbol{s}(\boldsymbol{x},0)=\boldsymbol{0}$, $\phi_1(\boldsymbol{x},0)=0$ and $\partial_t\phi_1(\boldsymbol{x},0)=0$. We seek to determine not only the temporal evolution of both $\boldsymbol{s}(\boldsymbol{x},t)$ and $\phi_1(\boldsymbol{x},t)$ for times $t\geqslant 0$, but also the incremental change $\delta\Omega$ in the axial rate of angular rotation. We note that $\boldsymbol{s}(\boldsymbol{x},t)$ and $\phi_1(\boldsymbol{x},t)$ are simply the infinitesimal elastic-gravitational response of the Earth model as seen by an observer in a reference frame rotating uniformly with constant angular velocity $\boldsymbol{\Omega}+\delta\boldsymbol{\Omega}=(\Omega+\delta\Omega)\,\hat{\boldsymbol{x}}_3$. As before, we will not consider explicitly the gravitational perturbation $\phi_1(\boldsymbol{x},t)$, since it can always be determined from $\boldsymbol{s}(\boldsymbol{x},t)$ by solving the incremental Poisson equation subject to the last two of the dynamical boundary conditions (65). Imperfections of elasticity in any physically realizable Earth model will always lead to the decay of all the non-secular normal modes of oscillation (except for the tilt-over mode but including the Chandler wobble). The final state of the Earth model, after the decay of all the non-secular modes, will be a new uniformly rotating equilibrium configuration with the same angular momentum vector $\boldsymbol{H}$, but with a generally different angular velocity of rotation $\boldsymbol{\Omega}+\delta\boldsymbol{\Omega}=(\Omega+\delta\Omega)\,\hat{\boldsymbol{x}}_3$. We will use $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$ to denote the infinitesimal displacement of the particle $\boldsymbol{x}$, measured in the frame of reference which is uniformly rotating with the angular velocity $\boldsymbol{\Omega}+\delta\boldsymbol{\Omega}$, and we will seek to determine $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$; relative to the frame of reference which rotates uniformly with angular velocity $\boldsymbol{\Omega}$, the Lagrangian particle position vector $\boldsymbol{r}(\boldsymbol{x},t)$ in the new static equilibrium configuration is given by

$$\boldsymbol{r}(\boldsymbol{x},t)=\boldsymbol{Q}(t)\cdot[\boldsymbol{x}+\boldsymbol{s}_{\text{final}}(\boldsymbol{x})].$$

We are now in a position to define precisely what is meant by a 'wholly internal' imposed body force distribution. We denote the Lagrangian description of the imposed body force density, measured per unit mass in the frame of reference rotating uniformly with angular velocity $\boldsymbol{\Omega}$, by $\boldsymbol{Q}(t)\cdot\boldsymbol{f}(\boldsymbol{x},t)$, and we suppose that $\boldsymbol{f}(\boldsymbol{x},t)$ may be prescribed; $\boldsymbol{f}(\boldsymbol{x},t)$ is then precisely the imposed body force density, as seen in the frame of reference which is rotating uniformly with the angular velocity $\boldsymbol{\Omega}+\delta\boldsymbol{\Omega}$. We suppose that after some finite time T, $\boldsymbol{f}(\boldsymbol{x},t)$ assumes the time-independent, possibly non-zero value $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$; thus $\boldsymbol{f}(\boldsymbol{x},t)=\boldsymbol{0}$ for $t<0$ and $\boldsymbol{f}(\boldsymbol{x},t)=\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ for $t\geqslant T$. The conditions that the imposed body force distribution exert neither a net force nor a net torque on the Earth model are, as before,

$$\left.\begin{aligned}\int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},t)]&=\boldsymbol{0},\\ \int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x})\,\boldsymbol{x}\times\boldsymbol{f}(\boldsymbol{x},t)]&=\boldsymbol{0}.\end{aligned}\right\}\tag{99}$$

Assuming $\Delta\boldsymbol{\Omega}(t)$ to be infinitesimal, the dynamical equations governing the elastic-gravitational response of a rotating Earth model to the imposed body force distribution $\boldsymbol{f}(\boldsymbol{x},t)$, for $t\geqslant 0$, can be shown to be

$$\left.\begin{aligned}\rho_0\partial_t^2\boldsymbol{s}+2\rho_0\boldsymbol{\Omega}\times\partial_t\boldsymbol{s}+\rho_0\partial_t\Delta\boldsymbol{\Omega}\times\boldsymbol{x}&\\ +\rho_0\boldsymbol{\Omega}\times(\Delta\boldsymbol{\Omega}\times\boldsymbol{x})+\rho_0\Delta\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}),&\\ =-\rho_0\boldsymbol{\nabla}\phi_1-\rho_0\boldsymbol{s}\cdot\boldsymbol{\nabla}[\boldsymbol{\nabla}(\phi_0+\psi)]+\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}}+\rho_0\boldsymbol{f},&\\ \nabla^2\phi_1=4\pi G\rho_1,&\\ \rho_1=-\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{s}),&\\ \tilde{\boldsymbol{T}}=\boldsymbol{\Lambda}:\boldsymbol{\nabla}\boldsymbol{s},&\end{aligned}\right\}\tag{100}$$

together with the dynamical free surface boundary conditions (65). The dynamical field equations (100) differ from the homogeneous dynamical field equations (64) not only by the presence

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of the inhomogeneous term $\rho_0(\mathbf{x})\mathbf{f}(\mathbf{x},t)$, but also by the presence of three terms linear in $\Delta\boldsymbol{\Omega}(t)$. We note that these equations are precisely the equations which would govern the elastic-gravitational response of the Earth model as viewed in the non-uniformly rotating reference frame with instantaneous angular velocity $\boldsymbol{\Omega}+\Delta\boldsymbol{\Omega}(t) = [\Omega+\delta\Omega H(t)]\,\hat{\mathbf{x}}_3$; we emphasize however that $\delta\Omega$ is as yet unknown, and we will maintain the point of view of an observer in the uniformly rotating reference frame with angular velocity $\boldsymbol{\Omega}$ by making use of the representation (97) of the elastic-gravitational deformation.

As in the non-rotating case, we shall make use of the Laplace transform (44), and we shall make the fundamental assumption that the complex normal mode displacement eigenfunctions $\mathbf{s}(\mathbf{x})$ of the Earth model form a complete linear space of complex vector-valued functions over V. We again let $\mathbf{s}_n(\mathbf{x})$, $1 \leqslant n \leqslant \infty$, be a set of basis eigenfunctions spanning this linear space. We are not in this case free to choose a set of basis eigenfunctions which are orthonormal in the sense (45), since displacement eigenfunctions of a rotating Earth model with distinct associated eigenfrequencies are not in general orthogonal; they satisfy only the quasi-orthogonality relation (81). We can and will choose normalized basis eigenfunctions, $(\mathbf{s}_n, \mathbf{s}_n) = \mathscr{T}(\mathbf{s}_n, \mathbf{s}_n) = 1$. Since the eigenfunctions $\mathbf{s}(\mathbf{x})$ of a rotating Earth model occur in complex conjugate pairs with an associated pair of eigenfrequencies $\pm\omega$, it is clear that the normalized basis eigenfunctions $\mathbf{s}_n(\mathbf{x})$ will necessarily occur in complex conjugate pairs as well (as noted above, there are two, but probably no more than two, exceptions to this rule, namely the eigenfunctions (38) and (40) corresponding to the axial translational and spin modes (88) and (89); neither of these exceptional basis eigenfunctions will play a role in the ensuing discussion).

We will assume that the Laplace transformed displacement field $\bar{\mathbf{s}}(\mathbf{x},p)$ and the final static displacement field $\mathbf{s}_{\text{final}}(\mathbf{x})$ and can be represented in the form

$$\left.\begin{aligned} \bar{\mathbf{s}}(\mathbf{x},p) &= \sum_{\substack{n=1\\ \omega_n\neq 0,\ \pm\Omega}}^{\infty} a_n(p)\,\mathbf{s}_n(\mathbf{x}),\\ \mathbf{s}_{\text{final}}(\mathbf{x}) &= \sum_{\substack{n=1\\ \omega_n\neq 0,\ \pm\Omega}}^{\infty} a_n^{\text{final}}\,\mathbf{s}_n(\mathbf{x}). \end{aligned}\right\} \tag{101}$$

We omit the three secular rigid body translational mode eigenfunctions, (38) and (39) with $\beta = 0$ and with $\alpha = 0$, as well as the two non-secular tilt-over mode eigenfunctions, (41) with $\beta = 0$ and with $\alpha = 0$, from the linear superpositions (101), because we have already shown that a body force distribution $\mathbf{f}(\mathbf{x},t)$ satisfying the constraints (99) cannot excite these modes. We likewise omit the single secular axial spin mode eigenfunction (40), in this case because we have incorporated the possible excitation of this mode into the change $\delta\Omega$ in the angular speed of rotation. We take the perfectly rigorous precaution of omitting these eigenfunctions

$$\{\mathbf{s}_n(\mathbf{x}) : \omega_n = 0,\ \pm\Omega\}$$

from the sums (101) *a priori* since, in contrast to the non-rotating case, the final analytical expressions we shall obtain for the associated $\{a_n(t) : \omega_n = 0,\ \pm\Omega\}$ and $\{a_n^{\text{final}} \neq 0,\ \pm\Omega\}$ are sufficiently complicated that we cannot as easily infer *a posteriori* that these coefficients must vanish. Note that, in the present instance, the lack of orthogonality among the basis eigenfunctions prevents any immediate description of the non-zero coefficients $a_n(p)$ and a_n^{final} in the representations (101) in terms of $\bar{\mathbf{s}}(\mathbf{x},p)$, $\mathbf{s}_{\text{final}}(\mathbf{x})$ and $\mathbf{s}_n(\mathbf{x})$, as in the first two of equations (47) in the non-rotating case. We insert the first of the representations (101) into the Laplace transformed version of the dynamical field equations (100), taking account of the time-dependence (98) of $\Delta\boldsymbol{\Omega}(t)$. We then form the inner product of the resultant equations with some particular basis eigenfunction, say $\boldsymbol{s}_m(\boldsymbol{x})$, where $\omega_m \neq 0, \pm\Omega$. Some applications of Gauss's theorem together with the boundary conditions (65) lead to

$$p^2 \sum_{\substack{n=1\\ \omega_n\neq 0,\,\pm\Omega}}^{\infty} a_n(\boldsymbol{s}_n, \boldsymbol{s}_m) - 2\mathrm{i}p \sum_{\substack{n=1\\ \omega_n\neq 0,\,\pm\Omega}}^{\infty} a_n \mathscr{W}(\boldsymbol{s}_n, \boldsymbol{s}_m) + \sum_{\substack{n=1\\ \omega_n\neq 0,\,\pm\Omega}}^{\infty} a_n[\mathscr{E}(\boldsymbol{s}_n, \boldsymbol{s}_m) + \mathscr{G}(\boldsymbol{s}_n, \boldsymbol{s}_m) + \Phi(\boldsymbol{s}_n, \boldsymbol{s}_m)] = f_m - (\delta\Omega/\Omega)\, g_m, \tag{102}$$

where we have simply defined

$$f_m(p) = (\bar{\boldsymbol{f}}, \boldsymbol{s}_m) = \int_V \mathrm{d}V\, [\rho_0(\boldsymbol{x})\bar{\boldsymbol{f}}(\boldsymbol{x},p)\cdot \boldsymbol{s}_m^*(\boldsymbol{x})] \tag{103}$$

and

$$g_m(p) = (\boldsymbol{\Omega}\times x + 2p^{-1}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times \boldsymbol{x}),\, \boldsymbol{s}_m) = \int_V \mathrm{d}V\, [\rho_0(\boldsymbol{x})\, \boldsymbol{\Omega}\times \boldsymbol{x}\cdot \boldsymbol{s}_m^*(\boldsymbol{x}) + 2p^{-1}\nabla\psi(\boldsymbol{x})\cdot \boldsymbol{s}_m^*(\boldsymbol{x})]. \tag{104}$$

Note that the terms linear in $\delta\Omega/\Omega$, and thus associated with the excitation of the axial spin mode, appear in the set of equations (102) as an apparent body force contribution. An appeal to either of equations (71) or (73), as well as to the quasi-orthogonality relation (81), can only reduce the system (102) to

$$a_m(p)\,[(p^2+\omega_m^2) - 2\mathrm{i}(p-\mathrm{i}\omega_m)\,(\boldsymbol{s}_m, \mathrm{i}\boldsymbol{\Omega}\times \boldsymbol{s}_m)] + \sum_{\substack{n=1\\ \omega_n\neq 0,\,\pm\Omega}}^{\infty} a_n(p)\,[(p-\mathrm{i}\omega_n)\,(p-i\omega_m)\,(\boldsymbol{s}_n, \boldsymbol{s}_m)] = f_m(p) - (\delta\Omega/\Omega)\, g_m(p). \tag{105}$$

This set of equations (105) is the proper generalization of the much simpler equations (49) to the rotating case; the effects of rotation are obviously profound. In the first place, the equations (102) contain the still unknown parameter $\delta\Omega/\Omega$; it is clear that the additional relation required for the determination of $\delta\Omega/\Omega$ can be obtained from a consideration of the law of conservation of angular momentum. In the second place, the equations (105) are an infinite set of simultaneous linear algebraic equations, and in general, all the transformed coefficients $a_m(p)$ will have to be simultaneously computed by solving these equations. The corresponding equations (49) in the non-rotating case are completely decoupled, and each $a_m(p)$ can be determined separately from a knowledge of $\bar{\boldsymbol{f}}(\boldsymbol{x},p)$ and the single associated complex displacement eigenfunction $\boldsymbol{s}_m(\boldsymbol{x})$.

Fortunately, we are able to make use of the theorem of residues to extract considerable information about the temporal coefficients $a_m(t)$ without having to find the full solution to the system of equations (105). We assume, as before, that each of the coefficients $f_m(p)$ is a regular function of the complex variable p, except at the origin $p = 0$, where each has a simple pole. We note that each of the coefficients $g_m(p)$ is similarly regular, except for a simple pole at $p = 0$. Equations (105) make it clear that each coefficient $a_m(p)$ will then have two simple poles, one at $p = 0$ and one at $p = \mathrm{i}\omega_m$. The pole of $a_m(p)$ at the origin will, as before, determine the ultimate static response $\boldsymbol{s}_{\mathrm{final}}(\boldsymbol{x})$ after the decay of all the normal modes of oscillation; we anticipate this by denoting the residue of $a_m(p)$ at $p = 0$ by a_m^{final}. The pole of $a_m(p)$ at $p = \mathrm{i}\omega_m$ will determine the dynamic or oscillatory part of the response $\boldsymbol{s}(\boldsymbol{x},t)$; the residue of $a_m(p)$ at $p = \mathrm{i}\omega_m$ may be obtained by multiplying equations (105) by $(p-\mathrm{i}\omega_m)$ and taking the limit as $p\to \mathrm{i}\omega_m$. This yields

$$\lim_{p\to \mathrm{i}\omega_m} [(p-\mathrm{i}\omega_m)\, a_m(p)] = \frac{f_m(\mathrm{i}\omega_m) - (\delta\Omega/\Omega)\, g_m(\mathrm{i}\omega_m)}{2\mathrm{i}[\omega_m + (\boldsymbol{s}_m, \mathrm{i}\boldsymbol{\Omega}\times \boldsymbol{s}_m)]}. \tag{106}$$

As before, we close the Bromwich contour in the right half p-plane for $t < 0$ and in the left half p-plane for $t \geqslant 0$; we may thus write $a_m(t)$ in the form

$$a_m(t) = \left\{a_m^{\text{final}} + \frac{1}{2}\left[\frac{f_m(\mathrm{i}\omega_m) - (\delta\Omega/\Omega)\, g_m(\mathrm{i}\omega_m)}{\mathrm{i}\omega_m + \mathrm{i}(\boldsymbol{s}_m, \mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{s}_m)}\right]\mathrm{e}^{\mathrm{i}\omega_m t}\right\} H(t), \tag{107}$$

and therefore

$$\boldsymbol{s}(\boldsymbol{x}, t) = \sum_{\substack{n=1\\ \omega_n \neq 0,\ \pm\Omega}}^{\infty} \left\{a_n^{\text{final}} + \frac{1}{2}\left[\frac{f_n(\mathrm{i}\omega_n) - (\delta\Omega/\Omega)\, g_n(\mathrm{i}\omega_n)}{\mathrm{i}\omega_n + \mathrm{i}(\boldsymbol{s}_n, \mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{s}_n)}\right]\mathrm{e}^{\mathrm{i}\omega_n t}\right\} \boldsymbol{s}_n(\boldsymbol{x})\, H(t). \tag{108}$$

Equation (108) completely determines the dynamic or oscillatory part of the displacement response $\boldsymbol{s}(\boldsymbol{x}, t)$, provided that the fractional change $\delta\Omega/\Omega$ in the angular rate of axial rotation is known. We note that, upon allowing for dissipation by replacing the eigenfrequencies ω_n in (108) by $\omega_n(1+\mathrm{i}/2Q_n)$, we find the long time limit of $\boldsymbol{s}(\boldsymbol{x}, t)$ to be precisely $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$, thereby justifying our notation for the residues at $p = 0$.

The static coefficients a_n^{final}, as well as the fractional change $\delta\Omega/\Omega$, in equation (108) are of course still unknown, and unfortunately their determination does, in general, require the solution of an infinite system of linear algebraic equations. As before, we will make use of the Tauberian theorems (51) and (54), presuming there to be some dissipative mechanism present in the Earth model in order to ensure the existence of the limit $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$. Multiplying equations (105) by p and taking the limit as $p \to 0$ yields

$$a_m^{\text{final}}[\omega_m^2 - 2\omega_m(\boldsymbol{s}_m, \mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{s}_m)] + \sum_{\substack{n=1\\ \omega_n \neq 0,\ \pm\Omega}}^{\infty} a_n^{\text{final}}[-\omega_n\omega_m(\boldsymbol{s}_n, \boldsymbol{s}_m)] = f_m^{\text{final}} - (\delta\Omega/\Omega)\, g_m^{\text{final}}, \tag{109}$$

where we have defined

$$f_m^{\text{final}} = (\boldsymbol{f}_{\text{final}}, \boldsymbol{s}_m) = \int_V \mathrm{d}V[\rho_0(\boldsymbol{x})\boldsymbol{f}_{\text{final}}(\boldsymbol{x})\cdot\boldsymbol{s}_m^*(\boldsymbol{x})] \tag{110}$$

and

$$g_m^{\text{final}} = 2(\boldsymbol{\nabla}\psi, \boldsymbol{s}_m) = 2\int_V \mathrm{d}V\,[\rho_0(\boldsymbol{x})\,\boldsymbol{\nabla}\psi(\boldsymbol{x})\cdot\boldsymbol{s}_m^*(\boldsymbol{x})]. \tag{111}$$

Conservation of angular momentum requires that the angular momentum of the new static equilibrium configuration be equal to that of the original equilibrium configuration. Let $\boldsymbol{C}+\delta\boldsymbol{C}$ be the inertia tensor of the new equilibrium configuration, as viewed by an observer rotating with that configuration, i.e. uniformly with angular velocity $\boldsymbol{\Omega}+\delta\boldsymbol{\Omega}$. Correct to first order in the small quantity $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$, we have

$$\delta\boldsymbol{C} = \int_V \mathrm{d}V\{\rho_0(\boldsymbol{x})\,[2\boldsymbol{x}\cdot\boldsymbol{s}_{\text{final}}(\boldsymbol{x}) - \boldsymbol{x}\boldsymbol{s}_{\text{final}}(\boldsymbol{x}) - \boldsymbol{s}_{\text{final}}(\boldsymbol{x})\,\boldsymbol{x}]\}. \tag{112}$$

The relation expressing the invariance of the angular momentum is

$$C\,\delta\Omega + (\hat{\boldsymbol{x}}_3\cdot\delta\boldsymbol{C}\cdot\hat{\boldsymbol{x}}_3)\,\Omega = 0,$$

or

$$\delta\Omega/\Omega = C^{-1}\sum_{\substack{n=1\\ \omega_n \neq 0,\ \pm\Omega}}^{\infty} a_n^{\text{final}}\int_V \mathrm{d}V\{\rho_0(\boldsymbol{x})\,[2\boldsymbol{x}\cdot\boldsymbol{s}_n(\boldsymbol{x}) - 2(\boldsymbol{x}\cdot\hat{\boldsymbol{x}}_3)\,(\boldsymbol{s}_n(\boldsymbol{x})\cdot\hat{\boldsymbol{x}}_3)]\}. \tag{113}$$

Equation (113) is a linear algebraic equation connecting $\delta\Omega/\Omega$ and the final static amplitudes a_m^{final}, and having coefficients which may be readily computed in terms of the basis eigenfunctions $\boldsymbol{s}_m(\boldsymbol{x})$. Adjoining the single equation (113) to the infinite linear system (109) produces a composite infinite system of linear algebraic equations whose simultaneous solution determines all of the static amplitudes a_m^{final}, and hence $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$ by means of the expansion (101), as well as the fractional change in rotation rate $\delta\Omega/\Omega$. Note that both $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$ and $\delta\Omega/\Omega$ depend only upon the final value $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ of the imposed body force distribution. As before, this is precisely because the final configuration must be exactly that static configuration which is in mechanical equilibrium with $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$. Knowledge of the static amplitudes a_m^{final}, as well as the fractional change in rotation rate $\delta\Omega/\Omega$ determines the complete displacement response $\boldsymbol{s}(\boldsymbol{x}, t)$ by means of equation (108). This rather cumbersome procedure must, in general, be employed in any exact computation of the infinitesimal elastic-gravitational response of a rotating Earth model to an imposed body force distribution. It is necessary to obtain a solution to the infinite coupled system of equations (109) and (113), and to determine thereby the full static response $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$ and $\delta\Omega/\Omega$, before the dynamic response (108) can be deduced. We note that the structure of equations (109) and (113) implies that if a_m^{final} is the final static amplitude corresponding to $\boldsymbol{s}_m(\boldsymbol{x})$, then $a_m^{\text{final}*}$ is the final static amplitude corresponding to the complex conjugate basis eigenfunction $\boldsymbol{s}_m^*(\boldsymbol{x})$, provided that $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ is real. This guarantees that the final displacement response $\boldsymbol{s}_{\text{final}}(\boldsymbol{x})$ to any real imposed body force distribution $\boldsymbol{f}_{\text{final}}(\boldsymbol{x})$ will be real, and, from equation (108), this in turn guarantees that the full displacement response $\boldsymbol{s}(\boldsymbol{x}, t)$ to any real $\boldsymbol{f}(\boldsymbol{x}, t)$ will be real.

There are a number of instances in which the above procedure may be circumvented, particularly if one is content to know only the dynamical or oscillatory part of the response $\boldsymbol{s}(\boldsymbol{x}, t)$. Geometrical symmetries of the Earth model, in particular, will generally give rise to a certain amount of decoupling of the infinite linear system (109) and (113). Consider, for example, the case of an Earth model which is axially symmetric about the axis of uniform rotation $\hat{\boldsymbol{x}}_3$. It is clear that every normal mode eigenfunction $\boldsymbol{s}_m(\boldsymbol{x})$ of such an axially symmetric Earth model can depend upon the azimuthal angle ϕ only through the combination $e^{i\mu\phi}$ where μ is an integer. It is easy to show, from equation (104), that the coefficient $g_m(p)$ associated with any eigenfunction $\boldsymbol{s}_m(\boldsymbol{x})$ will in that case be identically zero, except for those eigenfunctions (with $\mu = 0$) which are axially symmetric. This implies at once that the dynamical excitation amplitude of each of the $\mu \neq 0$ normal modes of oscillation may be computed disjointly, and without any knowledge of the eigenfunctions or eigenfrequencies of any other normal mode of the Earth model. This makes it possible, for example, to evaluate without difficulty, and in particular without having to solve an infinite linear system of algebraic equations, the extent to which the Chandler wobble of any axially symmetric Earth model (Smith 1974) may be excited by an earthquake source.

In the rotating limit $\boldsymbol{\Omega} \to \boldsymbol{0}$, equation (113) implies that $\delta\boldsymbol{\Omega} \to \boldsymbol{0}$ as well. Furthermore, the basis eigenfunctions $\boldsymbol{s}_n(\boldsymbol{x})$ become orthonormal and the infinite linear system (105) reduces to the simpler, completely decoupled non-rotating result (49). We see that the elegant and useful results given by Gilbert (1971) for the excitation of the elastic-gravitational normal modes of a non-rotating are a relatively simplified special case of the more general formulation appropriate to a rotating Earth model.

By far the most thoroughly studied normal modes of oscillation of the real Earth are the predominantly elastic normal modes, having characteristic periods of about one hour or less. Any Earth model which is everywhere solid and which is not on the verge of gravitational instability can have only two non-secular normal modes which do not fall into this class, namely the tilt-over mode (which cannot be excited by the type of body force distribution under consideration) and the Chandler wobble. The excitation of all the predominantly elastic modes by an imposed body force distribution can be effectively examined by making use of perturbation techniques. We assume that each of the predominantly elastic eigensolutions $\{\omega, \boldsymbol{s}(\boldsymbol{x}), \phi_1(\boldsymbol{x})\}$ of a rotating Earth model can be expressed in the form of a perturbation expansion

$$\left.\begin{aligned} \omega/\omega_0 &= \pm 1 + \sigma_1(\Omega/\omega_0) + \dots, \\ \boldsymbol{s}(\boldsymbol{x}) &= \boldsymbol{s}_0(\boldsymbol{x}) + \boldsymbol{s}_1(\boldsymbol{x})\,(\Omega/\omega_0) + \dots, \\ \phi_1(\boldsymbol{x}) &= \phi_1^0(\boldsymbol{x}) + \phi_1^1(\boldsymbol{x})\,(\Omega/\omega_0) + \dots, \end{aligned}\right\} \tag{114}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where $\{\omega_0, \mathbf{s}_0(\mathbf{x}), \phi_1^0(\mathbf{x})\}$ is an eigensolution of the normal mode boundary value problem (69) and (70) with both of the terms arising directly from rotation omitted, i.e.

$$\left.\begin{aligned} -\rho_0\omega_0^2\mathbf{s}_0 &= -\rho_0\boldsymbol{\nabla}\phi_1^0-\rho_0\mathbf{s}_0\cdot\boldsymbol{\nabla\nabla}\phi_0+\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{T}}_0,\\ \nabla^2\phi_1^0 &= 4\pi G\rho_1^0,\\ \rho_1^0 &= -\boldsymbol{\nabla}\cdot(\rho_0\mathbf{s}_0),\\ \tilde{\boldsymbol{T}}_0 &= \boldsymbol{\Lambda}:\boldsymbol{\nabla}\mathbf{s}_0, \end{aligned}\right\} \tag{115}$$

subject to the boundary conditions

$$\left.\begin{aligned} \hat{\boldsymbol{n}}\cdot\tilde{\boldsymbol{T}}_0 &= \mathbf{0},\\ [\phi_1^0]_-^+ &= 0,\\ [\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_1^0+4\pi G\rho_0\hat{\boldsymbol{n}}\cdot\mathbf{s}_0]_-^+ &= 0, \end{aligned}\right\} \tag{116}$$

on the free surface ∂V. Backus & Gilbert (1961) have justified the use of such an expansion (114), and have shown how conventional Rayleigh–Schrödinger perturbation theory can be used to deduce the various perturbations σ_1, $\mathbf{s}_1(\mathbf{x})$, $\phi_1^0(\mathbf{x})$, etc. Note that the zeroth-order eigensolution $\{\omega_0, \mathbf{s}_0(\mathbf{x}), \phi_1^0(\mathbf{x})\}$ will not in general be an eigensolution corresponding to any equilibrium non-rotating Earth model, since the static density and gravitational potential fields $\rho_0(\mathbf{x})$ and $\phi_0(\mathbf{x})$ which appear in (115) and (116) satisfy the mechanical equilibrium condition (57) and not (2). The form of equations (115) and (116) is however identical to that of (19) and (20), and the eigensolutions $\{\omega_0, \mathbf{s}_0(\mathbf{x}), \phi_1^0(\mathbf{x})\}$ will possess the same primitive algebraic properties as do those of any non-rotating Earth model. In particular, displacement eigenfunctions $\mathbf{s}_0(\mathbf{x})$ which are associated with distinct eigenfrequencies ω_0 will be orthogonal in the sense of the inner product (21). Furthermore, the eigensolutions $\{\omega_0, \mathbf{s}_0(\mathbf{x}), \phi_1^0(\mathbf{x})\}$ will occur naturally either in quartets associated with a two-dimensional primitive complex eigenspace, or in degenerate quartets associated with a one-dimensional primitive complex eigenspace. The perturbing effect of rotation is to induce a splitting of these primitive complex eigenspaces, either of the form (83) or (84).

We will consider explicitly only the case of a zeroth-order predominantly elastic eigenfrequency ω_0 which is reasonably well isolated in the zeroth-order eigenspectrum. Suppose that this zeroth-order eigenfrequency is associated with a non-degenerate zeroth-order quartet, with splitting of the form

$$\left(\begin{aligned} &\{\omega_0, \mathbf{s}_0(\mathbf{x}), \phi_1^0(\mathbf{x})\}\\ &\{-\omega_0, \mathbf{s}_0(\mathbf{x}), \phi_1^0(\mathbf{x})\}\\ &\{\omega_0, \mathbf{s}_0^*(\mathbf{x}), \phi_1^{0*}(\mathbf{x})\}\\ &\{-\omega_0, \mathbf{s}_0^*(\mathbf{x}), \phi_1^{0*}(\mathbf{x})\} \end{aligned}\right) \begin{matrix} \nearrow \begin{aligned} &\{\omega', \mathbf{s}'(\mathbf{x}), \phi_1'(\mathbf{x})\}\\ &\{-\omega', \mathbf{s}'^*(\mathbf{x}), \phi_1'^*(\mathbf{x})\} \end{aligned}\\ \searrow \begin{aligned} &\{\omega'', \mathbf{s}''(\mathbf{x}), \phi_1''(\mathbf{x})\}\\ &\{-\omega'', \mathbf{s}''^*(\mathbf{x}), \phi_1''^*(\mathbf{x})\}. \end{aligned} \end{matrix} \tag{117}$$

We choose $\mathbf{s}_0(\mathbf{x})$ and $\mathbf{s}_0^*(\mathbf{x})$ to be orthonormal in the sense (45). Degenerate Rayleigh–Schrödinger perturbation theory may be applied to this situation, with the result

$$\left.\begin{aligned} \omega'/\omega_0 &= 1+\sigma_1'(\Omega/\omega_0)+O(\Omega/\omega_0)^2,\\ \omega''/\omega_0 &= -1+\sigma_1''(\Omega/\omega_0)+O(\Omega/\omega_0)^2,\\ \mathbf{s}'(\mathbf{x}) = \mathbf{s}''(\mathbf{x}) &= \mathbf{s}_0(\mathbf{x})+O(\Omega/\omega_0),\\ \mathbf{s}'^*(\mathbf{x}) = \mathbf{s}''^*(\mathbf{x}) &= \mathbf{s}_0^*(\mathbf{x})+O(\Omega/\omega_0), \end{aligned}\right\} \tag{118}$$

where the eigenfrequency splitting parameters σ_1' and σ_1'' are given by (Backus & Gilbert 1961)

$$\sigma_1' = \sigma_1'' = (\mathbf{s}_0, \mathrm{i}\hat{\mathbf{x}}_3\times\mathbf{s}_0). \tag{119}$$

The inherently anisotropic influence of rotation on the generalized travelling wave normal modes of a rotating Earth model is apparent in the result (118), (119). If the associated zero-order quartet is degenerate, with splitting of the form

$$\left.\begin{matrix}\{\omega_0,\ \boldsymbol{s}_0(\boldsymbol{x}),\ \phi_1^0(\boldsymbol{x})\} \\ \{-\omega_0,\ \boldsymbol{s}_0(\boldsymbol{x}),\ \phi_1^0(\boldsymbol{x})\}\end{matrix}\right\} \longrightarrow \begin{matrix}\{\omega',\ \boldsymbol{s}'(\boldsymbol{x}),\ \phi_1'(\boldsymbol{x})\} \\ \{-\omega',\ \boldsymbol{s}'^*(\boldsymbol{x}),\ \phi_1'^*(\boldsymbol{x})\},\end{matrix} \tag{120}$$

then non-degenerate perturbation theory is applicable and we obtain

$$\left.\begin{aligned}\omega'/\omega_0 &= 1+O(\Omega/\omega_0)^2, \\ \boldsymbol{s}'(\boldsymbol{x}) &= \boldsymbol{s}_0(\boldsymbol{x})+O(\Omega/\omega_0), \\ \boldsymbol{s}'^*(\boldsymbol{x}) &= \boldsymbol{s}_0^*(\boldsymbol{x})+O(\Omega/\omega_0).\end{aligned}\right\} \tag{121}$$

The first-order correction to any non-degenerate eigenfrequency vanishes (Backus & Gilbert 1961).

It is a straightforward matter to express the displacement response $\boldsymbol{s}(\boldsymbol{x},t)$ associated with any well isolated, predominantly elastic, degenerate or non-degenerate quartet in terms of the perturbation-theoretical quantities ω_0, $\boldsymbol{s}_0(\boldsymbol{x})$, σ_1, etc. The general nature of the influence of rotation can be brought out clearly and simply by a consideration of the special case of an imposed body force distribution with a step function time-dependence, i.e.

$$\boldsymbol{f}(\boldsymbol{x},t) = \boldsymbol{f}_{\text{final}}(\boldsymbol{x})\,H(t) \quad \text{and} \quad \bar{\boldsymbol{f}}(\boldsymbol{x},p) = p^{-1}\boldsymbol{f}_{\text{final}}(\boldsymbol{x}).$$

We will also, for convenience, neglect the term in equation (108) which is linear in $\delta\Omega/\Omega$, since it does not affect any of the essential features which we wish to describe, and since it is likely to be either small or even identically zero in many situations of interest. Correct to zeroth-order in Ω/ω_0, the form of the particle displacement $\boldsymbol{s}(\boldsymbol{x},t)$ associated with a particular isolated non-degenerate split quartet of eigensolutions $\boldsymbol{s}'(\boldsymbol{x})$, $\boldsymbol{s}'^*(\boldsymbol{x})$, $\boldsymbol{s}''(\boldsymbol{x})$ and $\boldsymbol{s}''^*(\boldsymbol{x})$ may be written in the form

$$\boldsymbol{s}(\boldsymbol{x},t) = 2\,\mathrm{Re}\{[a_0^{\text{final}} - \omega_0^{-2} f_0^{\text{final}} \cos\omega_0 t\, \mathrm{e}^{\mathrm{i}\sigma_1\Omega t}]\,\boldsymbol{s}_0(\boldsymbol{x})\}H(t), \tag{122}$$

where

$$f_0^{\text{final}} = (\boldsymbol{f}_{\text{final}},\ \boldsymbol{s}_0) = \int_V \mathrm{d}V[\rho_0(\boldsymbol{x})\,\boldsymbol{f}_{\text{final}}(\boldsymbol{x})\cdot\boldsymbol{s}_0^*(\boldsymbol{x})]. \tag{123}$$

We see, by comparison of (122) with any two complex conjugate terms of the sum (56), that the standing wave pattern associated with any particular split quartet, which would remain stationary on a non-rotating Earth model, is induced to rotate backward, i.e. in a direction opposite to that of the rotation, at the angular rate $\sigma_1\Omega$ on a rotating Earth model. This slowly rotating standing wave pattern is clearly the result of superposing two travelling waves which have opposite directions and slightly different phase speeds. The form of the particle displacement $\boldsymbol{s}(\boldsymbol{x},t)$ associated with a particular isolated degenerate split quartet of eigensolutions $\boldsymbol{s}'(\boldsymbol{x})$ and $\boldsymbol{s}'^*(\boldsymbol{x})$ is simply, correct to zero order in Ω/ω_0,

$$\boldsymbol{s}(\boldsymbol{x},t) = 2\,\mathrm{Re}\{[a_0^{\text{final}} - \omega_0^{-2} f_0^{\text{final}} \cos\omega_0 t]\,\boldsymbol{s}_0(\boldsymbol{x})\}H(t). \tag{124}$$

The first-order effect of rotation on the associated eigenfrequencies is nil, and thus, correct to zeroth order, the associated particle motion (124) is a pure standing wave.

The above considerations are based on the premise that the zeroth-order predominantly elastic eigenfrequency under discussion is reasonably well isolated; these results may in general become invalid whenever two or more zeroth-order eigenfrequencies are closely spaced compared to Ω. Eigenfrequencies which are closely spaced compared to Ω must be treated together, by making use of quasi-degenerate Rayleigh–Schrödinger perturbation theory. The predominantly elastic eigenfrequencies of the real Earth occur in multiplets, each of which may be accurately regarded as arising from the splitting of the degenerate eigenfrequencies of the terrestrial monopole or spherically averaged Earth by the influence of rotation, asphericity, and anisotropy. The higher frequency multiplets are influenced much more strongly by the Earth's deviations from spherical symmetry than by its slow rotation (Luh 1974), and it is reasonable to expect that the above perturbation theoretical treatment of the excitation problem should be adequate, regardless of how closely spaced are the various eigenfrequencies within a given multiplet. The rates of rotation of the associated standing wave patterns are likely to be so low that they can probably be neglected for all observational purposes. The few very lowest frequency predominantly elastic multiplets are influenced more strongly by rotation than by the Earth's ellipticity of figure and other lateral heterogeneities. It should be a reasonable approximation in this case to utilize a fully spherically symmetric model of the Earth in order to compute the zeroth-order eigensolutions $\{\omega_0, \boldsymbol{s}_0(\boldsymbol{x}), \phi_1^0(\boldsymbol{x})\}$. This then allows the use of ordinary degenerate Rayleigh–Schrödinger perturbation theory rather than the quasi-degenerate variety in examining the perturbing effects of rotation. This case has been examined by Backus & Gilbert (1961). They showed that, correct to zero order in Ω/ω_0, the rotational splitting of a single degenerate multiplet has the character of Zeeman splitting, i.e. the perturbed eigenfrequencies are equally spaced from each other. This, together with an application of the addition theorem for surface spherical harmonics, is sufficient to show that the standing wave pattern $\boldsymbol{s}(\boldsymbol{x}, t)$ associated with the entire multiplet is induced to rotate backward at a uniform rate. We will not discuss in detail the most general case of quasi-degenerate zeroth-order eigenfrequencies, but the general nature of the influence of rotation on the predominantly elastic normal modes should nevertheless be clear.

4. Conclusions

The principal aim of this paper has been the construction of a general theoretical framework for examining the free elastic-gravitational normal modes of oscillation of a rotating Earth model. We have attempted, in so far as it is possible, to place this theoretical framework on a parallel basis with the somewhat more well-known theory of the free elastic-gravitational normal modes of a non-rotating Earth model. In the course of this, it was seen that not only is the theory for a non-rotating Earth model a special case of the more general theory for a rotating Earth model, it is a very much simpler, and consequently more elegant, special case. In particular, we have seen that the normal modes of oscillation of a rotating Earth model, unlike those of a non-rotating Earth model, cannot be composed as everywhere in-phase generalized standing waves; the inherently anisotropic influence of rotation causes every normal mode of oscillation of a rotating Earth model to be of the generalized travelling wave variety. Furthermore, the displacement eigenfunctions of any rotating Earth model are not in general mutually orthogonal, and they cannot be considered disjointly in seeking to determine at least the static part of the response to an imposed body force distribution.

We have at various points in this discussion restricted attention to only those rotating Earth models which are everywhere solid as well as gravitationally stable (and otherwise non-pathological). In particular, we did not discuss any of the complications which might arise from the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

presence of a fluid outer core and/or surficial oceans, although many of our results are probably most interesting and applicable in that case. It is well-known, for example, that any non-rotating Earth with a stably stratified fluid outer core can support not only an infinite suite of exactly zero-frequency modes corresponding to dilationless fluid flow along equipotential surface, but also a similarly infinite suite of very low frequency modes corresponding to internal buoyancy or gravity wave motions which will be essentially, but not completely confined to the fluid core. In §3.3, we described, via Rayleigh–Schrödinger perturbation theory, how the eigenfrequencies. which are associated with any pair of predominantly elastic oppositely directed generalized travelling waves are split apart by the influence of rotation by an amount which is of order Ω. In a remarkably similar fashion, as we discussed in § 3.2, the two zero-frequency equatorial rotational modes of any non-rotating Earth are split by rotation into two oppositely directed generalized travelling waves, whose associated eigenfrequencies are again separated by an amount of order Ω; in this latter instance, one of the associated eigenfrequencies has exactly the value $\pm\Omega$ (that associated with the tilt-over mode), while the other is very near to zero (that associated with the Chandler wobble). We conjecture that this phenomenon will be a fairly universal one, and that, in particular, the influence of rotation on the exactly zero-frequency and very low frequency fluid core modes will be to give rise to a suite of normal modes of oscillation with periods near one day and to another suite with very long periods. We note that precisely this behaviour is characteristic of the simplest constant vorticity motions of an incompressible fluid contained inside a rigid rotating ellipsoidal shell (Hough 1895; Poincaré 1910; Jeffreys & Vicente 1957*a*, *b*). In some as not yet understood sense, this phenomenon must give rise to the two large additional (i.e. not predominantly elastic) families of normal modes of oscillation which we expect to be associated with the presence of a stably stratified fluid core in a rotating Earth model, namely nearly diurnal internal inertial-gravity modes and long-period internal Rossby or planetary-gravity modes. A geophysically useful treatment of these various phenomena will require a great deal of further theoretical and numerical study.

We have also omitted any discussion of the elastic-gravitational response of either non-rotating or rotating Earth models to any externally imposed and harmonically varying tidal potential. The harmonic tidal response of any non-rotating Earth is reasonably well understood, but that of rotating Earth models, at least in the presence of a fluid outer core and/or oceans, is not. We contend that since the harmonic tidal response of any Earth model is bound to be most interesting at those frequencies which are near to any of the eigenfrequencies of the Earth model, a thorough study of the structure of the possible free oscillations should precede any study of the tidal response.

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References

Backus, G. E. & Gilbert, J. F. 1961 The rotational splitting of the free oscillations of the Earth. *Proc. natn. Acad. Sci., U.S.A.* **47**, 362–371.

Dahlen, F. A. 1973 Elastic dislocation theory for a self-gravitating elastic configuration with an initial static stress field. II. Energy release. *Geophys. J. R. astr. Soc.* **31**, 469–484.

Dahlen, F. A. 1974 On the static deformation of an Earth model with a fluid core. *Geophys. J. R. astr. Soc.* **36**, 461–485.

Eckart, C. 1960 *Hydrodynamics of oceans and atmospheres*. New York: Pergamon Press.

Edmonds, A. R. 1960 *Angular momentum in quantum mechanics*, § 2.5. Princeton University Press.

Gilbert, F. 1971 Excitation of the normal modes of the Earth by earthquake sources. *Geophys. J. R. astr. Soc.* **22**, 223–226.

Hough, S. S. 1895 The oscillations of a rotating ellipsoidal shell containing fluid. *Phil. Trans. R. Soc. Lond.* A **186**, 469–506.

Jeffreys, H. 1959 *The Earth* (4th ed.), arts 4.02, 4.03. Cambridge University Press.

Jeffreys, H. & Vicente, R. O. 1957*a* The theory of nutation and the variation of latitude, *Mon. Not. R. astr. Soc.* **117**, 142–161.

Jeffreys, H. & Vicente, R. O. 1957*b* The theory of nutation and the variation of latitude: the Roche model core. *Mon. Not. R. astr. Soc.* **117**, 162–173.

Lamb, H. 1932 *Hydrodynamics*, arts 202–206. Cambridge University Press. (Reprinted 1945 by Dover Publications, New York.)

Larmor, J. 1909 The relation of the Earth's free precessional nutation to its resistance against tidal deformation. *Proc. R. Soc. Lond.* A **82**, 89–96.

Love, A. E. H. 1909 The yielding of the Earth to disturbing forces. *Proc. R. Soc. Lond.* A **82**, 73–68.

Love, A. E. H. 1911 *Some principles of geodynamics*, arts 128–143. Cambridge University Press.

Luh, P. C. 1974 *Free oscillations of the laterally heterogeneous Earth*. Ph.D. thesis, University of California, La Jolla, California.

Lyttleton, R. A. 1953 *The stability of rotating liquid masses*, ch. II. Cambridge University Press.

Malvern, L. E. 1969 *Introduction to the mechanics of a continuous medium*, pp. 220–224. New Jersey: Prentice-Hall.

Munk, W. H. & MacDonald, G. J. F. 1960 *The rotation of the Earth*, § 3.2 and ch. 10. Cambridge University Press.

Poincaré, H. 1910 Sur la précession des corps déformables. *Bull. Astronomique* **27**, 321–356.

Rayleigh, Lord 1894 *The theory of sound*, vol. 1, arts 69–117, esp. art. 88. London: Macmillan. (Reprinted 1945 by Dover Publications, New York.)

Thomson, W. (Lord Kelvin) & Tait, P. G. 1897 *Treatise on natural philosophy*, art. 345. Cambridge University Press. (Reprinted 1962 as *Principles of mechanics and dynamics* by Dover Publications, New York.)

Smith, M. L. 1974 The scalar equations of infinitesimal elastic-gravitational motion for a rotating, slightly elliptical Earth. *Geophys. J. R. astr. Soc.* **37**, 491–526.

Whittaker, E. T. 1937 *A treatise on the analytical dynamics of particles and rigid bodies*, arts 76–86. Cambridge University Press. (Reprinted 1944 by Dover Publications, New York.)

Widder, D. V. 1946 *The Laplace transform*, ch. v. Princeton University Press.